APPM 2360 - Exam 1 February 21, 2001 , 7 - 8.30 pm

On the front of your blue book, write your name, the names of your lecturer (or lecture session number) and your TA (or recitation section number). Draw also a grading grid.

There are FIVE problems (some with subparts a, b, ...). YOU MUST WORK ALL FIVE PROBLEMS. Each full problem is worth 20 points. Start each problem on a new page. With the exception of problems 2a, 2b, and 5 (which require only the answers), show all your work in your bluebook. Box all your answers. Calculators, books or any notes are NOT permitted. No 'crib sheets' are allowed.

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1. Solve the following ODEs

   a. $x' + \frac{1}{9}x^2 = 1$.

   b. $x' = \frac{x}{t} + \frac{t}{x}$.

2. a. Write down Euler's formula for e^{ix} .

   b. Write $\cos x$ as a combination of exponentials.

   c. Express $(\cos x)^3$ as a combination of cosines of multiple angles (i.e. find the coefficients a_3, ... , a_0 so that $(\cos x)^3 = a_3 \cos 3x + a_2 \cos 2x + a_1 \cos x + a_0$).

3. Create a bifurcation diagram for the ODE $y' = (c^2 - y^2 - 1)(c - 2y)$ where c is a parameter.

   Be sure to mark all bifurcation points and indicate which branches are stable (attracting) and which are unstable (repelling). Label your axis clearly. Indicate with arrows where the solutions are increasing or decreasing.

4. Consider the ODE $x' = \frac{x}{t} - \frac{1}{2x}$.

   a. For which of the following initial conditions $x(t_0) = x_0$ does the existence and uniqueness theorem guarantee a unique solution of the ODE on some time-interval surrounding t_0? Motivate your answer clearly by using the theorem.

   i) $x(0) = 0$, ii) $x(0) = 1$, iii) $x(1) = 0$, iv) $x(1) = 1$.

   b. Find the general solution to the ODE.

   c. By using the general solution, determine how many solutions the ODE actually has when combined with each of the four initial conditions given in part a of this problem.

Please turn over ⇒

5. *Nine* different cases of first order ODEs $x' = f(t,x)$ were discussed in class (and in web notes) as permitting special solution methods. The forms of the nine ODE cases were as follows:

1. $x' = f(t)$
2. $x' = f(t) \cdot h(x)$
3. $x' = h\left(\frac{x}{t}\right)$
4. $x' = f(t) \cdot x + g(t)$
5. $x' = f(t) \cdot x + g(t) \cdot x^{\alpha}$
6. $x' = f(at + bx + c)$
7. $x' = \frac{a_1 x + b_1 t}{a_2 x + b_2 t}$
8. $x' = \frac{a_1 x + b_1 t + c_1}{a_2 x + b_2 t + c_2}$
9. $x' = \frac{f(a_1 x + b_1 t + c_1)}{g(a_2 x + b_2 t + c_2)}$

Below is a brief description of the main solution ideas for *five* of these cases:

a. *Homogeneous equation*. Test: change $\begin{cases} x & \to kx \\ t & \to kt \end{cases}$ and see if it made any difference. Set $u = \frac{x}{t} \Rightarrow x = t\,u \Rightarrow x' = u + t\,u'$; gives separable ODE for u.

b. *Bernoulli equation*. If $\alpha = 0$ or $\alpha = 1$, solve as before (linear or separable, resp.) Else, set $u = x^{1-\alpha}$, i.e. $x = u^{1/(1-\alpha)}, x' = \dots$ Gives a linear ODE for u.

c. *Separable equation*; write as $\int \frac{1}{h(x)} dx = \int f(t)\, dt$ and integrate.

d. If lines not parallel, find intersection (t_0, x_0), change variables $\hat{t} = t - t_0$, $\hat{x} = x - x_0$. Get homogeneous equation for $\hat{x}(\hat{t})$.

e. Direct integration (with respect to t).

Your task is to match up one of the equations with each of the five descriptions. Your answer (which for this problem requires no justifications) should take the form of a table as follows:

Description	a	b	c	d	e
Equation					

⇐ In each box in the second row, enter one of the numbers 1, 2, ... , 9