APPM 2360 Exam 1 Spring 2001

1. a, Normal form $x' = 1 - \frac{1}{9}x^2$

Separable: $\dfrac{x'}{1-\frac{1}{9}x^2} = 1$;

Partial fractions: $\dfrac{1}{1-\frac{1}{9}x^2} = \dfrac{-9}{x^2-9} = \dfrac{3/2}{x+3} - \dfrac{3/2}{x-3}$

So:

$$\frac{3}{2}\left[\frac{1}{x+3} - \frac{1}{x-3}\right]x' = 1,$$

$$\frac{3}{2}\ln\left|\frac{x+3}{x-3}\right| = t + c \Rightarrow \left|\frac{x+3}{x-3}\right| = e^{\frac{2t}{3}+\frac{2c}{3}} \Rightarrow$$

$$\frac{x+3}{x-3} = \underbrace{\pm e^{\frac{2c}{3}}}_{D \text{ constant}} \cdot e^{\frac{2t}{3}}$$

$$\Rightarrow x+3 = De^{\frac{2t}{3}}(x-3),$$

$$x(De^{2t/3}-1) = 3(De^{\frac{2t}{3}}+1), \quad \boxed{x(t) = 3\,\frac{De^{2t/3}+1}{De^{2t/3}-1}}$$

1 b, $x' = \frac{x}{t} + \frac{t}{x}$ 'Homogeneous'

Set $\frac{x}{t} = u \Rightarrow x = tu, \quad x' = u + tu'$

Plug into ODE gives

$$\cancel{u} + tu' = \cancel{u} + 1/u \Rightarrow uu' = \frac{1}{t}$$

Integrate: $\dfrac{u^2}{2} = \ln|t| + c$

$$u(t) = \pm\sqrt{\underbrace{2\ln|t|}_{\ln|t|^2 = 2\ln t} + \underbrace{2c}_{\text{constant } D}}$$

$$\boxed{x(t) = t \cdot u(t) = \pm t\sqrt{2\ln t + D}}$$

2a, $e^{ix} = \cos x + i \sin x$

b, From above, and

$e^{-x} = \cos x - i \sin x$

follows, after adding and divide by 2:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} .$$

c, $$(\cos x)^3 = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 = \frac{1}{8}\left(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}\right)$$

$$= \frac{1}{4}\left[\frac{e^{3ix}+e^{-3ix}}{2} + 3 \cdot \frac{e^{ix}+e^{-ix}}{2}\right] = \underline{\frac{1}{4}\cos 3x + \frac{3}{4}\cos x}$$

3a, $y' = f(c,y)$

$\uparrow (c^2-y^2-1)(c-2y)$

Solve $f(c,y)=0$.

$\Rightarrow \underbrace{c^2-y^2=1}$ or $\underbrace{c=2y}$

Hyperbola.

Or note that

$c^2 = 1+y^2$

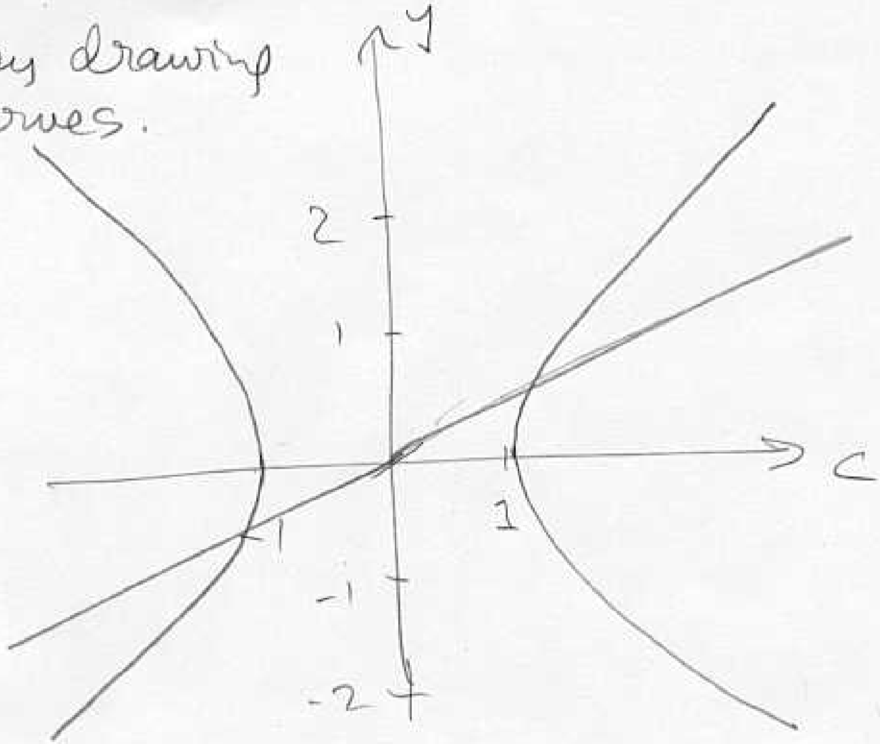
No solutions for $|c|<1$,

Otherwise

$y = \pm\sqrt{c^2-1}$.

Straight line through origin, slope ½ in (c,y) - plane

Start by drawing the curves.

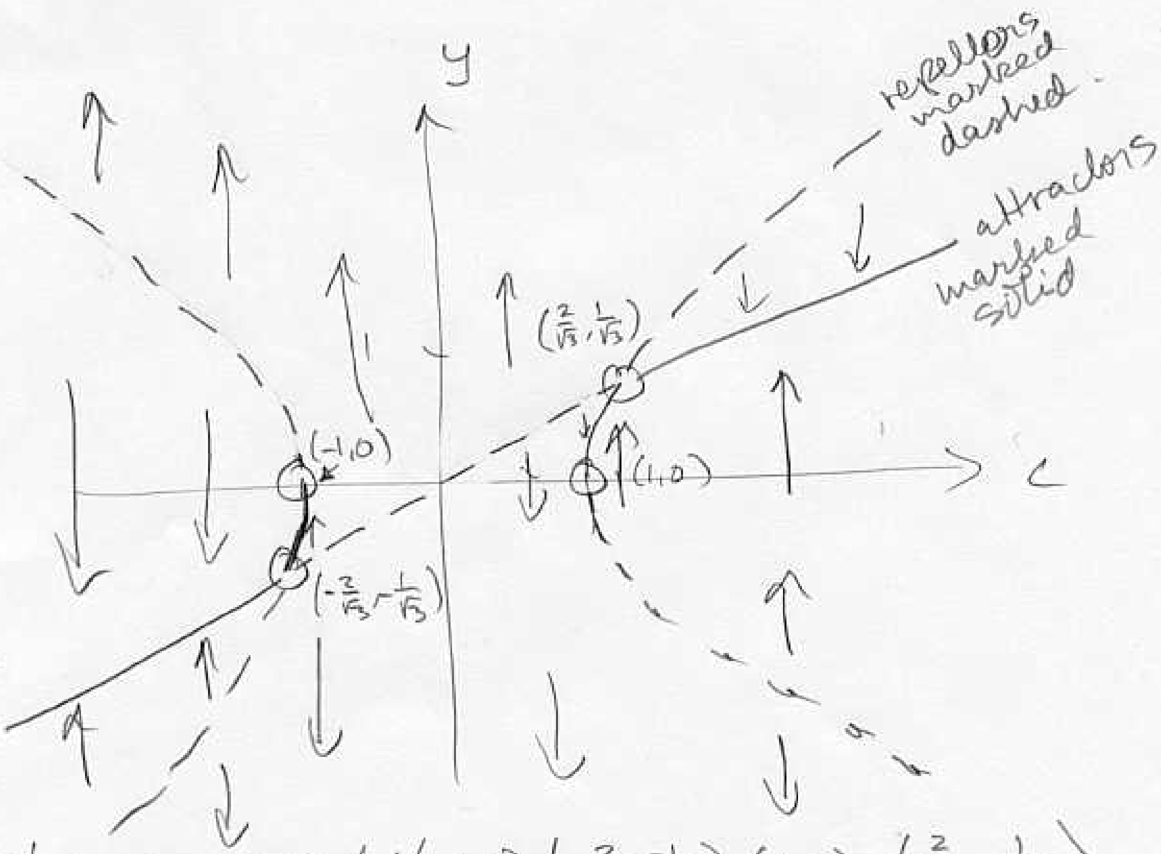


Check some signs:

For ex: $f > 0$ if y very large positive. arrow: ↑.

Swap sign as we cross any of the curves.

So fill in with arrows, mark attractors, repellers etc:



Four bifurcation points: $(-1,0)$, $(-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(1,0)$, $(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

4. a. $x' = f(t,x)$

$$\frac{x}{t} - \frac{1}{2x}.$$

f fails to be continuous when $t=0$ or $x=0$

$\frac{\partial f}{\partial x} = \frac{1}{t} + \frac{1}{2x^2}$: Same as above.

So we have existence and uniqueness in any rectangles that do not touch either of the t or x-axes.

So we are not guaranteed EU for the IC in i) ii) iii) but guaranteed for iv).

4 b, We recognize the ODE as a Bernoulli equation (see the type 5 and procedure b in problem 5).

$$x' = \frac{1}{t} \cdot x - \frac{1}{2} \cdot x^{-1} \qquad (1)$$

$\alpha = -1$.

Set $u = x^{1-(-1)} = x^2$,

$\Rightarrow u' = 2xx'$.

Multiply (1) by x : $xx' = \frac{x^2}{t} - \frac{1}{2}$

Becomes using u: $\frac{1}{2}u' = \frac{1}{t} \cdot u - \frac{1}{2}$,

$$u' = \frac{2}{t}u - 1$$

This is a linear equation; we can use variation of parameter or integrating factor

Using for ex. integrating factor, we get

$$u' - \frac{2}{t}u = -1$$

$p(t) = -\frac{2}{t}$, $P(t) = -2\ln t = \ln\frac{1}{t^2}$, $IF = e^{P(t)} = \frac{1}{t^2}$

Multiply by $\frac{1}{t^2}$,

$$\underbrace{\frac{1}{t^2}u' - \frac{2}{t^3}u}_{\frac{d}{dt}\left(\frac{u}{t^2}\right)} = -\frac{1}{t^2}$$

$$\frac{d}{dt}\left(\frac{u}{t^2}\right) = -\frac{1}{t^2}$$

$\Rightarrow \frac{u}{t^2} = \frac{1}{t} + c$, $\quad u = t + c\cdot t^2$

Return to x: $\quad x^2 = t + c\cdot t^2$

$$x(t) = \pm\sqrt{t + c\cdot t^2}$$

(Note: By chance, it happens that the substitution $u = \frac{x}{t}$ also works on the original ODE, although it is NOT homogeneous)

4 c, i, $x(0) = 0 \Rightarrow 0 = \pm\sqrt{0 + c\cdot 0}$

True for any value of $c \Rightarrow$ infinitely many solutions.

ii, $x(0) = 1 \Rightarrow 1 = \pm\sqrt{0 + c\cdot 0}$

Never true $\Rightarrow$ no solution

iii, $x(1) = 0 \Rightarrow 0 = \pm\sqrt{1 + c\cdot 1}$

Implies $c = -1$, but fails to distinguish between $+$ or $-$. So two solutions.

iv, $x(1) = 1 \Rightarrow 1 = \pm\sqrt{1 + c\cdot 1} \Rightarrow c = 0$ and "$+$".

Unique solution $x(t) = \sqrt{t}$.

5.

a	b	c	d	e
3	5	2	8	1
7				

Both answers correct on part a.