1)

1. $x'' = 2e^{x}, \quad x(0) = 0, \quad x'(0) = -2$

$$x' x'' = 2x' e^{x} \Rightarrow \int x' x'' dt = \int 2x' e^{x} dt$$

$$\Rightarrow \frac{1}{2}(x')^2 = 2\int \frac{dx}{dt} e^{x} dt = 2\int e^{x} dx = 2e^{x} + C_1$$

$$\Rightarrow \boxed{\frac{1}{2}(x')^2 = 2e^{x} + C_1}$$

I.C.

$$\frac{1}{2}(-2)^2 = 2e^{x(0)} + C_1$$

$$\frac{1}{2}\cdot 4 = 2 + C_1 \Rightarrow C_1 = 0$$

$$\Rightarrow (x')^2 = 4e^{x}$$

$$\Rightarrow x' = \pm 2e^{x/2}$$

But, since $x'(0) = -2 \Rightarrow$

$$\boxed{x' = -2e^{x/2}}$$

$$\frac{dx}{dt} = -2e^{x/2} \Rightarrow e^{-\frac{x}{2}} dx = -2 dt$$

$$\Rightarrow \int e^{-\frac{x}{2}} dx = -2\int dt$$

2)

$$-2e^{-\frac{x}{2}} = -2t + c_2$$

I.C. $\Rightarrow$ $-2 = c_2 \Rightarrow$

$$-2e^{-\frac{x}{2}} = -2t - 2$$

$$\Rightarrow e^{-\frac{x}{2}} = t + 1$$

$$\Rightarrow -\frac{x}{2} = \ln(t+1)$$

$$\Rightarrow \boxed{x = -2\ln(t+1)}$$

3)

$\boxed{2}$ $$x'' = \frac{4}{t}x' - \frac{4}{t^2}x$$

a) $x_1 = t$ is a solution: $x_1' = 1$, $x_1'' = 0$

$$\Rightarrow \quad 0 = \frac{4}{t} - \frac{4}{t^2}\cdot t = \frac{4}{t} - \frac{4}{t} = 0.$$

Therefore, $x_1 = t$ is indeed a solution.

b) To find a general solution, we set

$$x = u(t)\cdot x_1(t) = t\cdot u(t).$$

$$\Rightarrow \quad x' = u + tu' \Rightarrow \quad x'' = u' + u' + tu''$$

$$\Rightarrow \quad x'' = 2u' + tu''$$

$$\Rightarrow \quad 2u' + tu'' = \frac{4}{t}(u + tu') - \frac{4}{t^2}\cdot tu$$

$$2u' + tu'' = \cancel{\frac{4u}{t}} + 4u' - \cancel{\frac{4u}{t}}$$

$$\Rightarrow \quad tu'' = 2u'$$

$$\Rightarrow \quad \boxed{u'' = \frac{2}{t}u'}$$

4)

Now let $v = u' \Rightarrow v' = u''$

$$\Rightarrow \quad v' = \frac{2}{t} v$$

$$\Rightarrow \quad \frac{dv}{v} = \frac{2}{t} dt$$

$$\Rightarrow \quad \ln|v| = 2 \ln|t| + c_1$$

$$\Rightarrow \quad \ln|v| = \ln(c t^2)$$

$$\Rightarrow \quad v = c t^2$$

$$\Rightarrow \quad u' = ct^2 \Rightarrow \boxed{u = \frac{c}{3} t^3 + B}$$

$$\Rightarrow \boxed{X_2 = t \cdot u = \frac{c}{3} t^4 + Bt}$$

$\Rightarrow$ Here, we recover the former solution $X_1(t) = t$ and therefor, the second solution is $X_2(t) = t^4$

$$\Rightarrow \quad X(t) = c_1 X_1(t) + C_2 X_2(t) = c_1 t + C_2 t^4$$

$$\Rightarrow \quad \boxed{X = c_1 t + C_2 t^4}$$

5)

3.

a) $x'' + 4x = 0$

Let $x = e^{rt}$ $\Rightarrow$ $r^2 + 4 = 0 \Rightarrow r_1 = 2i,\ r_2 = -2i$

$\Rightarrow$ $\boxed{x(t) = c_1 \cos(2t) + c_2 \sin(2t)}$

b) $x'' + 4x = t$

Using the method of undetermined ceefficient, we guess,

$x = at + b$

$\Rightarrow$ $x' = a,\quad x'' = 0$

$\Rightarrow$ $4(at+b) = t \Rightarrow$ $4a = 1 \Rightarrow$ $\boxed{a = \frac{1}{4}}$

$4b = 0 \Rightarrow$ $\boxed{b = 0}$

$\Rightarrow$ $\boxed{x_p(t) = \frac{t}{4}}$

Therefore, the general solution is

$\boxed{x = c_1 \cos(2t) + c_2 \sin(2t) + \frac{t}{4}}$

6)

c) Using the method of variation of parameter,

we let

$$\boxed{x_p(t) = S_1(t)\cos(2t) + S_2(t)\sin(2t)} \quad (*)$$

Now, we have three options:

1) Plug $(*)$ into ODE, solve using

$$S_1' x_1 + S_2' x_2 = 0.$$

2) Use,

$$S_1' \cos(2t) + S_2' \sin(2t) = 0 \quad (1)$$

$$S_1' (-2\sin(2t)) + S_2' (2\cos(2t)) = t \quad (2)$$

3) Use,

$$S_1' = -\frac{r(t)\, x_2(t)}{W(t)}$$

$$S_2' = \frac{r(t)\, x_1(t)}{W(t)}.$$

Here, we use method (2).

From (1) we have $\boxed{S_1' = -S_2' \tan(2t)}$

Then substituting back into (2)

$$2S_2' \left(\cos(2t) + \frac{\sin^2(2t)}{\cos(2t)} \right) = t$$

$$\Rightarrow \quad S_2' = \frac{1}{2} t \cos(2t).$$

7)

but $\int t\cos(at)\,dt = \frac{\cos(at)}{a^2} + \frac{t\sin(at)}{a}$

$\Rightarrow \quad S_2 = \frac{1}{2}\int t\cos(2t)\,dt$

$\Rightarrow \quad S_2 = \frac{1}{2}\left[\frac{\cos(2t)}{4} + \frac{t\sin(2t)}{2}\right]$

$\Rightarrow \quad \boxed{S_2 = \frac{\cos(2t)}{8} + \frac{t\sin(2t)}{4}}$

$\Rightarrow$

Now, $S_1' = -S_2'\tan(2t) = -\frac{1}{2}t\cos(2t)\cdot\tan(2t)$

$\Rightarrow \quad S_1' = -\frac{1}{2}t\sin(2t)$

$\int t\sin(at)\,dt = \frac{\sin(at)}{a^2} - \frac{t\cos(at)}{a}$

$\Rightarrow \quad S_1 = -\frac{1}{2}\int t\sin(2t)\,dt = -\frac{1}{2}\left[\frac{\sin(2t)}{4} - \frac{t\cos(2t)}{2}\right]$

$\Rightarrow \quad \boxed{S_1 = -\frac{\sin(2t)}{8} + \frac{t\cos(2t)}{4}}$

8)

$$x_p(t) = S_1(t)\cos(2t) + S_2(t)\sin(2t)$$

$$\Rightarrow \quad x_p(t) = \left(-\frac{\sin(2t)}{8} + \frac{t\cos(2t)}{4}\right)\cdot\cos(2t)$$

$$+\left(\frac{\cos(2t)}{8} + \frac{t\sin(2t)}{4}\right)\sin(2t)$$

$$\Rightarrow \quad x_p(t) = -\cancel{\frac{\sin(2t)\cdot\cos(2t)}{8}} + \frac{t}{4}\cos^2(2t)$$

$$+\cancel{\frac{\cos(2t)\cdot\sin(2t)}{8}} + \frac{t}{4}\sin^2(2t)$$

$$\Rightarrow \quad x_p(t) = \frac{t}{4}\left(\cos^2(2t) + \sin^2(2t)\right)$$

$$\Rightarrow \quad \boxed{x_p(t) = \frac{t}{4}}$$

9)

$\boxed{4.}$

α) $x'' + x' + \frac{5}{4}x = 0$

$\Rightarrow \quad r^2 + r + \frac{5}{4} = 0$

$r = \dfrac{-1 \pm \sqrt{1 - 4 \cdot \frac{5}{4}}}{2} = \dfrac{-1 \pm \sqrt{-4}}{2}$

$\Rightarrow \quad \boxed{r_1 = -\frac{1}{2} + i \qquad r_2 = -\frac{1}{2} - i}$

spiral sink (e)

β) $x'' + \frac{x}{4} = 0 \Rightarrow r^2 = -\frac{1}{4} \Rightarrow r_1 = \frac{i}{2}, \; r_2 = -\frac{i}{2}$

center, (d)

γ) $x'' - \frac{7}{2}x' + \frac{3}{2}x = 0$

$r^2 - \frac{7}{2}r + \frac{3}{2} = 0$

$\Rightarrow \quad r = \dfrac{\frac{7}{2} \pm \sqrt{\frac{49}{4} - 4 \cdot \frac{3}{2}}}{2} = \dfrac{7}{4} \pm \dfrac{\sqrt{\frac{49}{4} - \frac{24}{4}}}{2}$

$r = \frac{7}{4} \pm \frac{1}{2}\sqrt{\frac{25}{4}} \Rightarrow r_1 = \frac{7}{4} + \frac{5}{4} = \frac{12}{4} = 3$

10)

$$r_2 = \frac{7}{4} - \frac{5}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \boxed{\begin{array}{ll} r_1 = 3, & r_2 = \frac{1}{2} \\ \text{Source} & (c) \end{array}}$$

10 δ) $x'' + \frac{5}{2}x' - \frac{3}{2}x = 0.$

$$r^2 + \frac{5}{2}r - \frac{3}{2} = 0$$

$$r = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} + \frac{12}{2}}}{2}$$

$$r_1 = -\frac{5}{4} + \frac{1}{2}\sqrt{\frac{49}{4}} = -\frac{5}{4} + \frac{1}{2}\frac{7}{2} = -\frac{5}{4} + \frac{7}{4} = \frac{1}{2}$$

$$r_2 = -\frac{5}{4} - \frac{1}{2}\sqrt{\frac{49}{4}} = -3$$

$$\Rightarrow \boxed{\begin{array}{ll} r_2 = -3, & r_1 = \frac{1}{2} \\ \text{Saddle} & (a) \end{array}}$$

11)

ε) $$x'' + \frac{7}{2}x' + \frac{3}{2}x = 0$$

$$r^2 + \frac{7}{2}r + \frac{3}{2} = 0.$$

$$r = \frac{-\frac{7}{2} \pm \sqrt{\left(\frac{7}{2}\right)^2 - 4\cdot\frac{3}{2}}}{2}$$

$$\Rightarrow \quad r_1 = \left(-\frac{7}{2} + \sqrt{\frac{49}{4} - \frac{24}{4}}\right)\cdot\frac{1}{2} = \frac{1}{2}\left(-\frac{7}{2} + \frac{5}{2}\right)$$

$$r_1 = -\frac{1}{2}, \quad r_2 = \frac{-\frac{7}{2} - \sqrt{\frac{49}{4} - \frac{24}{4}}}{2} = -3$$

$$\Rightarrow \quad \boxed{r_1 = -\frac{1}{2},\ r_2 = -3}$$

Sink (b)

Description	Figure	ODE
Sink	b	ε
Source	c	γ
Saddle	a	δ
Spiral sink	e	α
Center.	d	β