Solutions, Exam 3, Spring 2001

1. $Ax=0$ has non-trivial solutions $\iff \det(A)=0$

$$\begin{vmatrix} a & 1 & \frac{3}{2} \\ -2 & a & 0 \\ 0 & a & 1 \end{vmatrix} = -a\begin{vmatrix} a & \frac{3}{2} \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} a & 1 \\ -2 & a \end{vmatrix} = -3a + a^2 + 2$$

$$a^2 - 3a + 2 = 0 \iff a = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}} = \frac{3}{2} \pm \frac{1}{2}$$

$$\boxed{a=2 \text{ or } a=1}$$

Alternative solution:

Perform Gaussian elimination on $Ax=0$

$$\left(\begin{array}{ccc|c} a & 1 & \frac{3}{2} & 0 \\ -2 & a & 0 & 0 \\ 0 & a & 1 & 0 \end{array}\right) \rightleftarrows \left(\begin{array}{ccc|c} -2 & a & 0 & 0 \\ a & 1 & \frac{3}{2} & 0 \\ 0 & a & 1 & 0 \end{array}\right) \frac{a}{2}$$

$$\left(\begin{array}{ccc|c} -2 & a & 0 & 0 \\ 0 & 1+\frac{a^2}{2} & \frac{3}{2} & 0 \\ 0 & a & 1 & 0 \end{array}\right) -\frac{a}{1+a^2/2} \quad \left(\begin{array}{ccc|c} -2 & a & 0 & 0 \\ 0 & 1+\frac{a^2}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1-\frac{3/2\, a}{1+a^2/2} & 0 \end{array}\right)$$

There can be a nonzero solution only if

$$1 - \frac{3/2\, a}{1+a^2/2} = 0 \iff 1 + \frac{a^2}{2} - \frac{3}{2}a = 0 \iff$$

$$a^2 - 3a + 2 = 0 \implies$$

$$\boxed{a=2 \text{ or } a=1}$$

2. To solve $\bar{x}' = A\bar{x}$, find eigenvalues and eigenvectors.

a) $A = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10$$

$$\lambda^2 - 2\lambda + 10 = 0 \iff \lambda = 1 \pm \sqrt{1-10} = 1 \pm 3i$$

$$\Rightarrow \boxed{\lambda_1 = 1+3i \quad \lambda_2 = 1-3i}$$

$\underline{\lambda_1 = 1+3i:}$ $\left(\begin{array}{cc|c} 1-3i & 5 & 0 \\ -2 & -1-3i & 0 \end{array}\right) \xrightarrow{\frac{1+3i}{5}} \left(\begin{array}{cc|c} 1-3i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $v_2 = s \Rightarrow (1-3i)v_1 + 5s = 0 \iff$

$$v_1 = -\frac{5}{1-3i}s = -\frac{5(1+3i)}{10}s = \left(-\frac{1}{2} - \frac{3}{2}i\right)s$$

$\bar{v} = \begin{pmatrix} -\frac{1}{2} - \frac{3}{2}i \\ 1 \end{pmatrix} s$ Choose for ex. $\bar{v}_1 = \begin{pmatrix} 1+3i \\ -2 \end{pmatrix}$

The second eigenvector $\bar{v}_2$ is the complex conjugate of $\bar{v}_1$

$$\Rightarrow \boxed{\bar{v}_1 = \begin{pmatrix} 1+3i \\ -2 \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} 1-3i \\ -2 \end{pmatrix}}$$

2 contd. Find the real valued solutions from the complex valued solution $z = \bar{v}_1 e^{\lambda_1 t}$

$$z = \begin{pmatrix} 1+3i \\ -2 \end{pmatrix} e^{(1+3i)t} = \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) e^t (\cos 3t + i \sin 3t) =$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t \cos 3t - \begin{pmatrix} 3 \\ 0 \end{pmatrix} e^t \sin 3t + i \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t \sin 3t + \begin{pmatrix} 3 \\ 0 \end{pmatrix} e^t \cos 3t \right]$$

$$\bar{x}_1 = \operatorname{Re}(z) = \begin{pmatrix} e^t (\cos 3t - 3 \sin 3t) \\ -2 e^t \cos 3t \end{pmatrix}$$

$$\bar{x}_2 = \operatorname{Im}(z) = \begin{pmatrix} e^t (\sin 3t + 3 \cos 3t) \\ -2 e^t \sin 3t \end{pmatrix}$$

General solution:

$$\boxed{\bar{x}(t) = c_1 \bar{x}_1 + c_2 \bar{x}_2 = c_1 \begin{pmatrix} e^t (\cos 3t - 3 \sin 3t) \\ -2 e^t \cos 3t \end{pmatrix} + c_2 \begin{pmatrix} e^t (\sin 3t + 3 \cos 3t) \\ -2 e^t \sin 3t \end{pmatrix}}$$

2b) $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)+1 =$

$= \lambda^2 - 4\lambda + 4$

$\lambda^2 - 4\lambda + 4 = 0 \iff \lambda = 2 \pm \sqrt{4-4} = 2$

$\Rightarrow \boxed{\lambda_1 = 2 \ , \ \lambda_2 = 2}$

$\underline{\lambda = 2:}$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $v_2 = s \Rightarrow v_1 + s = 0 \Rightarrow v_1 = -s$

$\bar{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$ Choose $\boxed{\bar{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Only one eigenvector, deficient eigenspace

Find generalized eigenvector

$(A-2I)\bar{u} = \bar{v}_1 \iff \left(\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$

Let $u_2 = s \Rightarrow u_1 + s = -1 \Rightarrow u_1 = -1-s$

$\bar{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$ Choose $\boxed{\bar{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$

$\bar{x}_1 = \bar{v}_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}$ $\quad \bar{x}_2 = (\bar{u} + t\bar{v}_1)e^{\lambda_1 t} = \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) e^{2t}$

Gen. sol. $\boxed{\bar{x}(t) = c_1\bar{x}_1 + c_2\bar{x}_2 = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} -1-t \\ t \end{pmatrix} e^{2t}}$

2c) For both a) and b) both eigenvalues have a positive real part

$\Rightarrow$ hyperbolic source

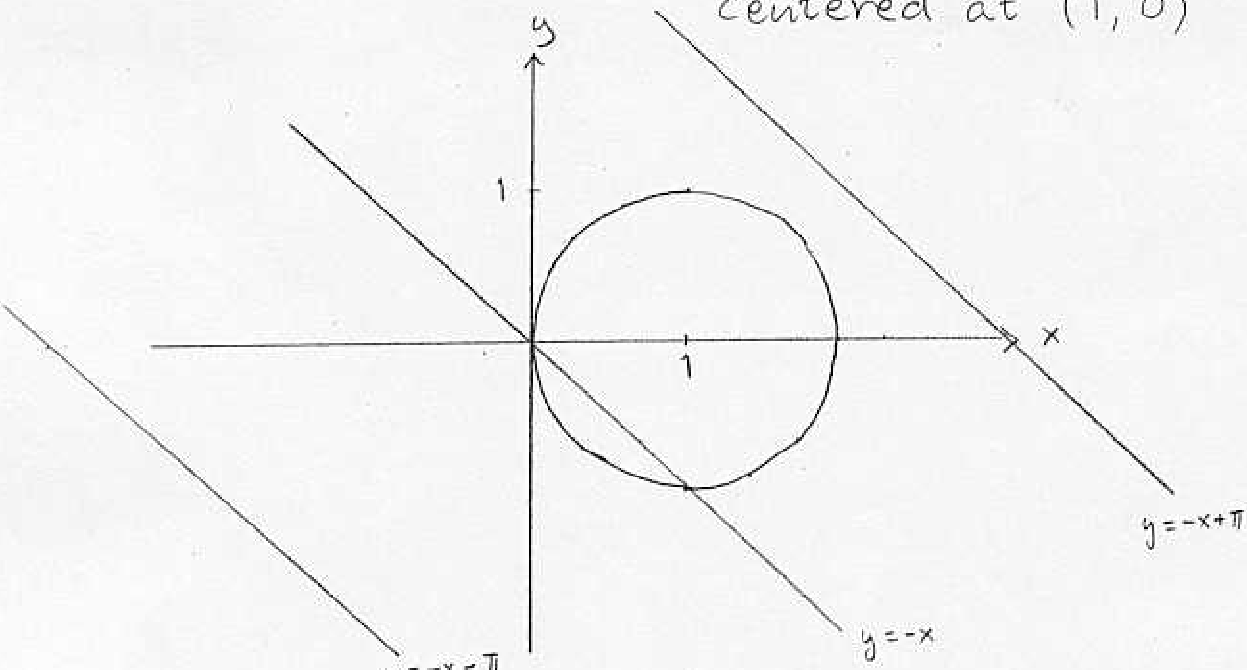
3. $\begin{cases} x' = \sin(x+y) \\ y' = (x-1)^2 + y^2 - 1 \end{cases}$

a) Find the two equilibrium points

$$0 = \sin(x+y) \qquad (1)$$
$$0 = (x-1)^2 + y^2 - 1 \qquad (2)$$

$(1) \Rightarrow \quad x + y = n\pi \quad \Leftrightarrow \quad y = -x + n\pi$

$(2) \Rightarrow \quad (x-1)^2 + y^2 = 1$ Circle of radius one centered at $(1, 0)$



The equilibria are found for $y = -x$

Insert into (2) $\Rightarrow \quad (x-1)^2 + x^2 - 1 = 0$

$$\Leftrightarrow \quad 2x^2 - 2x = 0 \quad \Leftrightarrow \quad 2x(x-1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 1$$

$\boxed{\text{Equilibria } (0,0) \text{ and } (1,-1)}$

3a, <u>Alternative</u> solution:

$$0 = \sin(x+y) \qquad (1)$$
$$0 = (x-1)^2 + y^2 - 1 \qquad (2)$$

$$(1) \Rightarrow \quad x + y = n\pi \quad \Leftrightarrow \quad y = -x + n\pi$$

Insert into (2)

$$x^2 - 2x + 1 + x^2 - 2xn\pi + n^2\pi^2 - 1 = 0 \Leftrightarrow$$
$$2x^2 - (1+n\pi)2x + n^2\pi^2 = 0 \qquad \Leftrightarrow$$
$$x^2 - (1+n\pi)x + \frac{n^2\pi^2}{2} = 0 \qquad \Leftrightarrow$$
$$x = \frac{1+n\pi}{2} \pm \sqrt{\frac{1+2n\pi+n^2\pi^2}{4} - \frac{2n^2\pi^2}{4}} \qquad \Leftrightarrow$$
$$x = \frac{1+n\pi}{2} \pm \frac{\sqrt{1+2n\pi-n^2\pi^2}}{2}$$

Real solutions if $\quad 1 + 2n\pi \geq n^2\pi^2$

True for $n=0$ only

$$\Rightarrow \qquad x = \frac{1}{2} \pm \frac{1}{2}$$

$$\Rightarrow \qquad x = 1 \quad \text{and} \quad x = 0$$

$$\boxed{\text{Equilibria } (0,0) \text{ and } (1,-1)}$$

36, Linearize around $(0,0)$ (No shift needed)

$$\begin{cases} x' = \sin(x+y) = f(x,y) \\ y' = (x-1)^2 + y^2 - 1 = g(x,y) \end{cases}$$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ 2x-2 & 2y \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$$

Linearized system $\boxed{\bar{x}' = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \bar{x}}$

<u>or</u>

Linearize around $(1,-1)$. Needs shifting

Let $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} u_1 + 1 \\ u_2 - 1 \end{pmatrix}$

$$\begin{cases} u_1' = \sin(u_1 + u_2) = f(u_1, u_2) \\ u_2' = u_1^2 + (u_2 - 1)^2 - 1 = g(u_1, u_2) \end{cases}$$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} \\ \frac{\partial g}{\partial u_1} & \frac{\partial g}{\partial u_2} \end{pmatrix} = \begin{pmatrix} \cos(u_1+u_2) & \cos(u_1+u_2) \\ 2u_1 & 2u_2 - 2 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$$

Linearized system $\boxed{\bar{u}' = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \bar{u}}$

4.

	True	False
a		F
b		F
c		F
d	T	
e		F