SOLUTION APPM 2360 FINAL. May 8, 2001.

1. Solution by integrating factor method:

$$x' + \underbrace{\frac{2}{t}}_{p(t)} x = \frac{\cos t}{t}$$

$$p(t) = \frac{2}{t} \Rightarrow P(t) = 2\ln t, \quad IF = e^{P(t)} = e^{2\ln t} = t^2$$

so

$$\underbrace{t^2 x' + 2tx}_{\frac{d}{dt}(t^2 \cdot x)} = t\cos t$$

$$\Rightarrow t^2 x = \int t\cos t\, dt = \begin{pmatrix}\text{integration}\\ \text{by parts}\end{pmatrix} = \int t\, d(\sin t) =$$

$$= t\sin t - \int \sin t\, dt = t\sin t + \cos t + C$$

$$\Rightarrow x(t) = \frac{1}{t^2}(t\sin t + \cos t + C)$$

$$IC \Rightarrow \frac{1}{\pi^2} = \frac{1}{\pi^2}(\underbrace{\pi \sin\pi}_{=0} + \underbrace{\cos\pi}_{=-1} + C) \Rightarrow C = 2$$

So $\boxed{x(t) = \frac{1}{t^2}(t\sin t + \cos t + 2)}$

2 Find the general solution of

$$\frac{dx}{dt} = \sin^2(3t - 3x + 1)$$

Solution Let $u = 3t - 3x + 1$

$$\Rightarrow \quad \frac{du}{dt} = 3 - 3\frac{dx}{dt}$$

$$\Rightarrow \quad 3\frac{dx}{dt} = 3 - \frac{du}{dt} \Rightarrow \boxed{\frac{dx}{dt} = 1 - \frac{1}{3}\frac{du}{dt}}$$

$$\Rightarrow \quad 1 - \frac{1}{3}\frac{du}{dt} = \sin^2 u \Rightarrow \quad 1 - \sin^2(u) = \frac{1}{3}\frac{du}{dt}$$

$$\Rightarrow \quad \frac{1}{3}\frac{du}{dt} = \cos^2 u$$

$$\Rightarrow \quad \frac{du}{\cos^2 u} = 3dt$$

$$\Rightarrow \quad \int \frac{du}{\cos^2 u} = 3\int dt \Rightarrow \quad \tan(u) = 3t + C$$

$$\Rightarrow \quad \tan(3t - 3x + 1) = 3t + C$$

$$\Rightarrow \quad 3t - 3x + 1 = \arctan(3t + c)$$

$$\Rightarrow \quad \boxed{\frac{3t - \arctan(3t + C) + 1}{3} = x(t)}$$

3. Find the general solution to

$$x'' + x = \frac{1}{\cos t}$$

Solution

First, we solve the associated homogeneous equation:

$$x'' + x = 0$$

$$\Rightarrow \quad r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$$

$$\Rightarrow \quad x_h(t) = c_1 \cos t + c_2 \sin t.$$

Next, to solve the non-homogeneous equation, we use the method of variation of parameter. we look for a solution in the form

$$x_p(t) = c_1(t)\, x_1(t) + c_2(t)\, x_2(t)$$

where, $x_1(t) = \cos t$ and $x_2(t) = \sin t$

$$\Rightarrow \quad \boxed{x_p(t) = c_1(t) \cos t + c_2(t) \sin t}$$

Then c_1 and c_2 are obtained from:

$$c_1' = -\frac{r(t)\,x_2(t)}{W(t)}$$

$$c_2' = +\frac{r(t)\,x_1(t)}{W(t)}$$

where $W(t)$ is the Wronskian given by

$$W(t) = x_1 x_2' - x_1' x_2 \qquad \text{and} \qquad r(t) = \frac{1}{\cos t}.$$

$\Rightarrow \quad x_1' = -\sin t, \qquad x_2' = \cos t$

$\Rightarrow \quad W(t) = \cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad \boxed{W = 1}$

$c_1' = -\frac{1}{\cos t} \cdot \sin t \quad \Rightarrow \quad c_1' = -\tan(t).$

$\Rightarrow \quad c_1 = -\int \frac{\sin t}{\cos t}\,dt = \ln|\cos t|$

$c_2' = \frac{1}{\cos t} \cdot \cos t \quad \Rightarrow \quad c_2' = 1 \Rightarrow \boxed{c_2 = t}$

$\Rightarrow \boxed{x(t) = c_1 \cos t + c_2 \sin t + (\cos t)\ln|\cos t| + t \sin t}$

4.

Find the general solution of

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

solution

We first find the eigenvalues and eigenvectors of A where

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}.$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 2 & -3 \\ -2 & 4-\lambda & -3 \\ -2 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} - 2 \begin{vmatrix} -2 & -3 \\ -2 & -1-\lambda \end{vmatrix} - 3 \begin{vmatrix} -2 & 4-\lambda \\ -2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow -\lambda\left[-(\lambda+1)(4-\lambda) + 6\right] - 2\left[2(1+\lambda) - 6\right] - 3\left[-4 + 2(4-\lambda)\right] = 0$$

$$-\lambda\left[\lambda^2 - 3\lambda + 2\right] - 2\left[2 + 2\lambda - 6\right] - 3\left[-4 + 8 - 2\lambda\right] = 0$$

$$-\lambda(\lambda-1)(\lambda-2) - 4(\lambda-2) + 6(\lambda-2) = 0.$$

$$\Rightarrow \quad (\lambda-2)\left[-\lambda^2+\lambda-4+6\right] = 0$$

$$\Rightarrow \quad -(\lambda-2)(\lambda^2-\lambda-2) = 0$$

$$\Rightarrow \quad -(\lambda-2)(\lambda-2)(\lambda+1) = 0$$

$$\Rightarrow \quad \boxed{(\lambda-2)^2(\lambda+1) = 0}$$

$$\boxed{\lambda_1 = -1, \qquad \lambda_{2,3} = 2.}$$

Finding eigenvectors:

For $\lambda = -1$

We have:

$$\begin{pmatrix} 1 & 2 & -3 \\ -2 & 5 & -3 \\ -2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

row #2 − row #3

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & -3 \\ -2 & 5 & -3 \\ 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$ $2 \cdot \text{row}\ \#1 + \text{row}\ \#2$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 9 & -9 \\ 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} x_1 + 2x_2 - 3x_3 = 0 \\ x_2 - x_3 = 0. \end{matrix}\right\} \Rightarrow \begin{matrix} x_2 = x_3 \\ x_1 = 3x_3 - 2x_2 \end{matrix}$$

choose, $x_3 = s \Rightarrow x_2 = s \Rightarrow x_1 = 3s - 2s = s.$

$$\Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

<u>For $\lambda = 2$</u>

we have:
$$\begin{pmatrix} -2 & 2 & -3 \\ -2 & 2 & -3 \\ -2 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -2 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{-2x_1 + 2x_2 - 3x_3 = 0}$$

we choose, $x_3 = s_1$ and $x_2 = s_2$

$$\Rightarrow \quad 2x_1 = 2x_2 - 3x_3 = 2s_2 - 3s_1$$

$$\Rightarrow \quad x_1 = s_2 - \frac{3}{2}s_1$$

$$\Rightarrow \vec{v} = \begin{bmatrix} s_2 - \frac{3}{2}s_1 \\ s_2 \\ s_1 \end{bmatrix} = \begin{bmatrix} s_2 \\ s_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2}s_1 \\ 0 \\ s_1 \end{bmatrix}$$

$$= s_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Finally, the solution will read:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} e^{2t}}$$

5. Find a general solution by using series expansion to

(*) $$x'' - 2tx' - 4x = 0.$$

Solution

We let $$x = \sum_{n=0}^{\infty} a_n t^n$$

Then, $$x' = \sum_{n=1}^{\infty} n a_n t^{n-1}, \quad x'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

substituting in (*) we get,

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - 2\sum_{n=1}^{\infty} n a_n t^n - 4\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Rightarrow \boxed{\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - 2k a_k - 4a_k\right] t^k = 0}$$

We got the above relation by replacing n by $k+2$ in the first summation.

Then,

$$(k+2)(k+1)a_{k+2} - 2k a_k - 4a_k = 0, \qquad k = 0, 1, 2, \cdots$$

$$\Rightarrow \quad a_{k+2} = \frac{2(k+2)}{(k+1)(k+2)} a_k \qquad k = 0, 1, 2, 3, \cdots$$

$$\Rightarrow \quad \boxed{a_{k+2} = \frac{2}{k+1} a_k}$$

For even integers.

$k=0,$ $\quad a_2 = 2a_0$

$k=2,$ $\quad a_4 = \frac{2}{3} a_2 = \frac{2^2}{3} a_0$

$k=4,$ $\quad a_6 = \frac{2}{5} a_4 = \frac{2^3}{1 \cdot 3 \cdot 5} a_0$

$k=6,$ $\quad a_8 = \frac{2}{7} a_6 = \frac{2^4}{1 \cdot 3 \cdot 5 \cdot 7} \cdot a_0$

$$\Rightarrow \quad \boxed{a_{2n} = \frac{2^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} a_0} \qquad n = 1, 2, \cdots$$

For odd integers:

$k=1,$ $\quad a_3 = a_1$

$k=3,$ $\quad a_5 = \frac{2}{4} a_3 = \frac{1}{1 \cdot 2} a_1$

$k=5, \quad a_7 = \frac{2}{6} a_5 = \frac{1}{1\cdot 2\cdot 3} a_1$

$k=7, \quad a_9 = \frac{2}{8} a_7 = \frac{1}{1\cdot 2\cdot 3\cdot 4} a_1$

$$\Rightarrow \quad \boxed{a_{2n+1} = \frac{1}{n!} a_1}$$

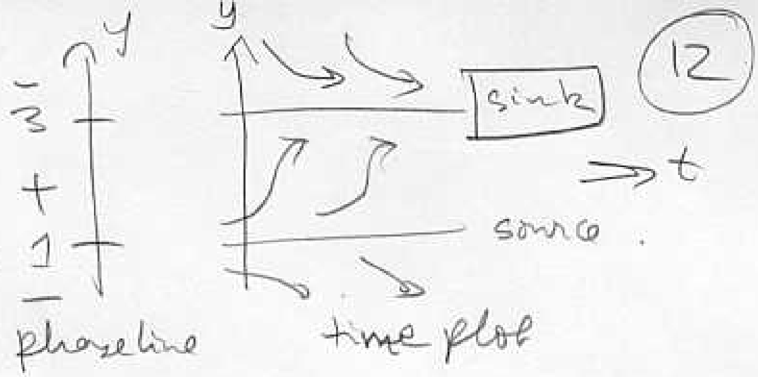
$$\Rightarrow \quad x(t) = a_0 \left[1 + 2t^2 + \frac{2^2}{1\cdot 3} t^4 + \frac{2^3}{1\cdot 3\cdot 5} t^6 + \cdots + \frac{2^n}{1\cdot 3\cdot 5 \cdots (2n-1)} t^{2n} \right]$$

$$+ a_1 \left[t + t^3 + \frac{1}{2!} t^5 + \frac{1}{3!} t^7 + \cdots + \frac{1}{n!} t^{2n+1} \right]$$

$$\Rightarrow \quad \boxed{x(t) = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{2^n}{1\cdot 3\cdot 5 \cdots (2n-1)} t^{2n} \right] + a_1 \left[\sum_{n=0}^{\infty} \frac{1}{n!} t^{2n+1} \right]}$$

Since $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, we recognize the last sum as $t \sum_{n=0}^{\infty} \frac{1}{n!} t^{2n} = \boxed{te^{t^2}}$.

6. a. The signs of the right hand side $-y^2+4y-3$ are as shown →



b. $x' = F(t,x) = t\sqrt{x}$ continuous

$\frac{\partial F}{\partial x} = \frac{1}{2}\frac{t}{\sqrt{x}}$ <u>not</u> continuous at $x=0$.

$\boxed{\text{So EU will not tell anything around } x(1)=0}$

C. $c_1 \cos t + c_2 \sin t \equiv 0 \Rightarrow$ Take derivative $-c_1 \sin t + c_2 \cos t \equiv 0$.

Multiply by $\sin t$ and $\cos t$ resp. and add

$\Rightarrow c_2 \cdot 1 = 0 \Rightarrow c_2 = 0 \quad \Rightarrow c_1 = 0$.

So linearly $\boxed{\text{independent}}$

d. General solution of a second order linear homogeneous ODE is a combination of <u>two</u> independent solutions.

e. In general, (for $x' = f(t,x)$)

$x_{n+1}(t) = x_0(t) + \int_0^t f(s, x_n(s))\,ds$

Here $x_0 \equiv 1$, so

$x_1(t) = 1 + \int_0^t 1^2\,dt = \underline{1+t}$

	TRUE	FALSE
a		F
b		F
c	T	
d		F
e		F