On the front of your bluebook, write your name and your instructor's name, and make a score-chart.

There are **FIVE** questions. **YOU MUST WORK ALL FIVE PROBLEMS**. Start each problem on a new page, show **ALL** your work in your bluebook and $\boxed{\text{box}}$ in your answers. Calculators, books and crib sheets are **NOT** permitted.

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1. In this problem, we consider the use of Gaussian Elimination (row reduction) for solution of a matrix equation of the form

$$A\mathbf{x} = \mathbf{b}\,. \tag{1}$$

(a) (i) Under what condition does a unique solution to (1) exist?

(ii) Under what condition does (1) have either no solution or infinitely many solutions ?

(iii) For the case of infinitely many solutions we write the solution as

$$\mathbf{x} = r\mathbf{x}_0 + \mathbf{x}_1\,, \tag{2}$$

where r can be any real number. What matrix equations are satisfied by the vectors $\mathbf{x}_0$ and $\mathbf{x}_1$?

(b) Let us define matrix A and vector $\mathbf{b}$ in (1) as

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 1 & 0 \end{bmatrix}, \qquad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ c \\ -1 \end{bmatrix}.$$

There is a unique value of c such that (1) has infinitely many solutions, whereas for all other values of c the equation has no solution. What is this value of c?

(c) Having found c such that (1) has infinitely many solutions, solve (1) with this value for c in vector $\mathbf{b}$, and express the solution $\mathbf{x}$ in the form (2).

2. Consider the system of ODEs

$$\mathbf{x}' = A\mathbf{x}\,. \tag{3}$$

(a) If A is a defective matrix, with a double eigenvalue λ_1 that has only one eigenvector $\mathbf{v_1}$,

(i) Show that $\mathbf{x_1} = \mathbf{v_1}\exp(\lambda_1\,\mathbf{t})$ is a solution of (3).

(ii) Derive the condition that vector $\mathbf{u_1}$ has to satisfy such that $\mathbf{x_2} = (\mathbf{u_1} + \mathbf{t}\,\mathbf{v_1})\exp(\lambda_1\,\mathbf{t})$ is also a solution of (3).

(b) Given the (non-defective) matrix

$$A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix},$$

find the general solution of (3). Express your final result in terms of real functions.

3. Consider the system of ODEs

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(\mathbf{t}), \tag{4}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}(\mathrm{t}) = \begin{bmatrix} \cos(t) \\ \exp(3\,t) \end{bmatrix}.$$

Two linearly independent solutions of the homogeneous equation are given by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(t) \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp(-\mathrm{t}).$$

   (a) Write down a form for a particular solution of the equation that is appropriate for use in the method of the undetermined coefficients.

   (b) Find the particular solution.

4. For matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

find all the eigenvalues and a complete set of linearly independent eigenvectors and generalized eigenvectors if necessary.

5. Answer **TRUE** or **FALSE** to the following question; no justification is required.

   (a) If matrix $\mathbf{A}$ has n distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$, then the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

   (b) Suppose λ_1 is an eigenvalue of multiplicity m for matrix A, where m is a positive integer, $m \geq 1$. That is, the characteristic polynomial is $|A - \lambda I| = (\lambda - \lambda_1)^m q(\lambda)$, where $q(\lambda_1) \neq 0$. Then A always has m linearly independent eigenvectors associated with λ_1.

   (c) If the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) + c_3\mathbf{w}(t_0) = \mathbf{0}$ is satisfied only for $c_1 = c_2 = c_3 = 0$, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vector functions, defined at some t_0 in the interval $I = [a, b]$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent on I.

   (d) If the matrix elements of $\mathbf{A}(t)$ are continuous for all $t \in [t_1, t_3]$, and the vector elements of $\mathbf{b}(t)$ are continuous for all $t \in [t_2, t_4]$, with $t_1 < t_2 < t_3 < t_4$, then the general solution to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$ exists for all $t \in [t_2, t_3]$.

   (e) If $|\mathbf{A}| = 0$ then the only equilibrium solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x} = \mathbf{0}$.