1. In this problem we consider the use of Gaussian Elimination (row reduction) for solution of a matrix equation of the form

$$A\mathbf{x} = \mathbf{b}\,. \tag{1}$$

(a) (i) A unique solution to (1) exists if and only if $\det(A) \neq 0$, or if the columns of A are linearly independent.

(ii) Equation (1) has either no solution or infinitely many solutions if and only if $\det(A) = 0$, or if the columns of A are linearly dependent.

(iii) For the case of infinitely many solutions we write the (general) solution as

$$\mathbf{x} = r\mathbf{x}_0 + \mathbf{x}_1\,, \tag{2}$$

where r can be any real number. Let us find the matrix equations satisfied by vectors $\mathbf{x}_0$ and $\mathbf{x}_1$. To accomplish this, substitute (2) into (1) to obtain

$$A\mathbf{x} = A(r\mathbf{x}_0 + \mathbf{x}_1) = r(A\mathbf{x}_0) + A\mathbf{x}_1\,,$$

and so we must have

$$r(A\mathbf{x}_0) + A\mathbf{x}_1 = \mathbf{b}\,. \tag{3}$$

Now observe that, since r is arbitrary, we can choose $r = 0$ in (3), which implies $A\mathbf{x}_1 = \mathbf{b}$. This is the matrix equation satisfied by vector $\mathbf{x}_1$. We can now subtract $A\mathbf{x}_1$ from the left-hand side of (3) and $\mathbf{b}$ from the right-hand side, since they are equal, to find $r(A\mathbf{x}_0) = \mathbf{0}$. Since this must hold for *any* value of r, we can choose $r = 1$, which implies $A\mathbf{x}_0 = \mathbf{0}$, which is the matrix equation satisfied by vector $\mathbf{x}_0$.

(b) With matrix A and right-hand side vector $\mathbf{b}$ as defined, we begin the Gaussian elimination procedure with the following augmented matrix,

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 3 & 1 & 2 & c \\ 5 & 1 & 0 & -1 \end{array}\right].$$

Add $-3/4$ times the 1st row to the 2nd, and $-5/4$ times the 1st row to the 3rd, to obtain

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 1/4 & 5/4 & c-3 \\ 0 & -1/4 & -5/4 & -6 \end{array}\right].$$

Now add 1 times the 2nd row to the 3rd row to obtain

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 1/4 & 5/4 & c-3 \\ 0 & 0 & 0 & c-9 \end{array}\right].$$

At this point, the 3rd equation in our set (3rd row) states that $0 = c - 9$. If this equality is not true, then there is no solution, so to have solutions we must have $c = 9$.

(c) Continue the row reduction with $c = 9$. Multiply the 2nd row by 4 to obtain

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 1 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now add -1 times the 2nd row to the 1st row to obtain

$$\left[\begin{array}{ccc|c} 4 & 0 & -4 & -20 \\ 0 & 1 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Finally multiply the 1st row by 1/4 to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -5 \\ 0 & 1 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have now reduced the system as far as is possible, having manipulated the rows until we have ones on the main diagonal wherever possible, with zeroes above and below these diagonal entries. The three equations in three unknowns represented by this augmented matrix are

$$\begin{aligned} x_1 - x_3 &= -5 \\ x_2 + 5x_3 &= 24 \\ 0 &= 0 \end{aligned}$$

Thus, we can write $x_1 = -5 + x_3$, $x_2 = 24 - 5x_3$, and let x_3 be a "free parameter." Define $x_3 = r$, then our solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r-5 \\ -5r+24 \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 24 \\ 0 \end{bmatrix}.$$

2. Consider the system of ODEs

$$\mathbf{x}' = A\mathbf{x}. \tag{4}$$

(a) Suppose that A has a double eigenvalue λ, which has only one eigenvector $\mathbf{v}$.

(i) We first show that $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$ is a solution. We have $\mathbf{x}_1' = \lambda e^{\lambda t}\mathbf{v}$, and $A\mathbf{x}_1 = e^{\lambda t}A\mathbf{v} = e^{\lambda t}\lambda\mathbf{v}$. Hence $\mathbf{x}_1' = A\mathbf{x}_1$.

(ii) (see the discussion accompanying equations (5) and (6) on p. 181 of Diacu, and exercise 17 on p.187) For a second linearly independent solution, try an expression of the form $\mathbf{x}_2 = e^{\lambda t}(\mathbf{u} + t\mathbf{v})$, where $\mathbf{u}$ is an unknown, but constant, vector. Let us derive the condition that vector $\mathbf{u}$ must satisfy so that $\mathbf{x}_2$ is also a solution of (4). To do this, plug $\mathbf{x}_2$ into (4) and see where it will lead us. We have

$$\mathbf{x}_2' = \lambda e^{\lambda t}(\mathbf{u} + t\mathbf{v}) + e^{\lambda t}\mathbf{v},$$

and also

$$A\mathbf{x}_2 = e^{\lambda t}(A\mathbf{u} + t(A\mathbf{v})) = e^{\lambda t}(A\mathbf{u} + t(\lambda\mathbf{v})),$$

since $A\mathbf{v} = \lambda\mathbf{v}$. Now $\mathbf{x}_2' = A\mathbf{x}_2$ if and only if

$$\begin{aligned} \lambda e^{\lambda t}(\mathbf{u} + t\mathbf{v}) + e^{\lambda t}\mathbf{v} &= e^{\lambda t}(A\mathbf{u} + t\lambda\mathbf{v}) \\ \lambda\mathbf{u} + \lambda t\mathbf{v} + \mathbf{v} &= A\mathbf{u} + t\lambda\mathbf{v} \\ \lambda\mathbf{u} + \mathbf{v} &= A\mathbf{u} \\ \mathbf{v} &= A\mathbf{u} - \lambda\mathbf{u} \\ (A - \lambda I)\mathbf{u} &= \mathbf{v}, \end{aligned}$$

which is the condition on $\mathbf{u}$. Note that $\mathbf{u}$ is what we call a "generalized eigenvector."

(b) Now we solve (4) with $A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$. The characteristic polynomial is $|A - \lambda I| = \lambda^2 - 2\lambda + 5$, which equals zero if $\lambda = 1 \pm 2i$. Choose $\lambda = 1 + 2i$ and find an eigenvector $\mathbf{v}$, by solving

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Adding $-2i$ times the 1st row to the 2nd row wipes out the 2nd row, and we have

$$\begin{bmatrix} -2i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $v_2 = -2iv_1$, so choose $v_1 = 1$, and an eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -2iv_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

To write the general solution, we build two linearly independent solutions from the real and imaginary parts of our eigenvalue and eigenvector, as it is done in the text (p.182 of Diacu) or in class. The general solution is $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ with

$$\begin{aligned} \mathbf{x}_1(t) &= e^t \left\{ \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}, \\ \mathbf{x}_2(t) &= e^t \left\{ \cos 2t \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \end{aligned}$$

and where c_1 and c_2 are arbitrary constants (slightly different answer is possible).

3. Consider the non-homogeneous system of ODEs

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos t \\ e^{3t} \end{bmatrix}. \qquad (5)$$

We are given that two solutions to the associated homogeneous equation are

$$\mathbf{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and asked to find a particular solution $\vec{\phi}(t)$ which will acount for the right-hand side vector $\mathbf{b}$.

(a) Following the rules for constructing such particular solutions (see top of p.175 of Diacu), we choose the following as our trial solution,

$$\vec{\phi}(t) = e^{3t} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \cos t \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \sin t \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, \qquad (6)$$

where the various coefficients are, as of yet, undetermined. Note that we built this trial solution from two different pieces, one to account for the exponential function and a second one to account for the trigonometric function.

(b) To solve, we substitute (6) into (5) and find the various relationships which will allow us to determine values for our coefficients. We find three matrix equations, by matching all matrix/vector terms belonging with $e^{3t}, \cos t, \sin t$, respectively, which are

$$\begin{aligned} A \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} &= 3 \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ A \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} &= \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} &= - \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \end{aligned}$$

where A is the matrix in (5). The matrix being sufficiently "easy" we can just write out the six equations in six unknowns, which are

$$\begin{aligned} b_0 &= 3a_0\,, & b_1 &= a_2 - 1\,, & b_2 &= -a_1\,, \\ a_0 &= 3b_0 - 1\,, & a_1 &= b_2\,, & a_2 &= -b_1\,. \end{aligned}$$

Solving these, we find the particular solution

$$\vec{\phi}(t) = e^{3t}\begin{bmatrix} 1/8 \\ 3/8 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} + \sin t \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

4. For the given matrix A, we are asked to find a complete set of linearly independent vectors and generalized eigenvectors, if necessary. It is important to understand that a "complete set" must consist of 3 vectors, for a 3×3 matrix, and that we will need to find a generalized eigenvector(s) only if there are fewer than three different eigenvectors. First compute

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3\,,$$

so $\lambda = 1$ is our only eigenvalue, with multiplicity 3 (that is, repeated three times). To find eigenvectors, set $\lambda = 1$ and solve

$$(A - I)\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We obtain a single relationship from this matrix equation, which is $x_2 + x_3 = 0$, from which we can express x_2 in terms of x_3 by writing $x_2 = -x_3$. Now our eigenvector is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

where x_1 and x_3 are free parameters, and the two constant vectors are linearly independent eigenvectors. Any sum of these two is also an eigenvector, so that this is a general expression which accounts for *all* eigenvectors of A belonging to $\lambda = 1$.

Now, since we only got two linearly independent eigenvectors, and we need three vectors for a complete set, we must construct a generalized eigenvector $\mathbf{u}$, which must satisfy $(A - I)\mathbf{u} = \mathbf{x}$, which in matrix form is

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix}.$$

Note that, for this system to have a solution, we must set $x_3 = 0$, then we have $u_2 + u_3 = x_1$, so

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ x_1 - u_3 \\ u_3 \end{bmatrix} = u_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1 \\ 0 \end{bmatrix}.$$

This vector $\mathbf{u}$ is a generalized eigenvector for *any* choice of u_1 and u_2, but it also depends on our choice of x_1 in the eigenvector. Note also that the eigenvectors we found above turned up again in this expression for $\mathbf{u}$. The reason the eigenvectors showed up here is the same reason we saw the vector $\mathbf{x}_0$ which satisfies $A\mathbf{x}_0 = \mathbf{0}$ in problem 1(a)iii, can you figure out why?

In our general expression for $\mathbf{u}$, take $x_1 = 1$, and $u_1 = u_2 = 0$, to get

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{then our complete set is} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

5. True/False questions:

(a) True—eigenvectors belonging to distinct (different) eigenvalues are linearly independent (see paragraph at the bottom of p.165 of Diacu).

(b) False—we have seen, in problem 4 for example, that we don't always get a number of linearly independent eigenvectors equal to the multiplicity of the eigenvalue.

(c) True—this is a direct consequence of the theorem (see Theorem 4.2.2 on p. 172 of Diacu) which states that if the Wronskian $W(t)$ is non-zero for at least one value of t on the interval I, then the functions are linearly independent on I. Since $W(t)$ is the determinant of a matrix whose columns are the vector functions under investigation, it follows that if $c_1 = c_2 = c_3 = 0$ is the only (unique) solution of $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) + c_3\mathbf{w}(t_0) = \mathbf{0}$, then $W(t_0) \neq 0$ (see Theorem 4.1.1 on p. 162 of Diacu).

(d) True—we just need both $A(t)$ and $\mathbf{b}(t)$ continuous on the same interval (see Existence and Uniqueness Theorem on p. 170 of Diacu).

(e) False—equilibrium solutions occur if $A\mathbf{x} = \mathbf{0}$, and there will always be an infinity of them if $|A| = 0$. This is easily proved by noting that if $|A| = 0$ then there will be either no solution or infinitely many solutions to the equation $A\mathbf{x} = \mathbf{b}$. But with $\mathbf{b} = \mathbf{0}$ we can see immediately that $\mathbf{x} = \mathbf{0}$ is a solution, hence it isn't "no solution," so it must be infinitely many (see also Remark 2 at the top of p.163 of Diacu).