

## Exam 1 Solutions

1. Consider the ordinary differential equation (ODE)

$$\frac{x'}{2 \sin t \cos t} = x.$$

- a. Write the equation in normal form.
- b. Determine any singular solutions.
- c. Use separation of variables to determine the general solution.
- d. Determine the particular solution when  $x(0) = 1$ .

### SOLUTION

- a. The equation in normal form is

$$x' = 2x \sin t \cos t.$$

- b. Singular solutions are solutions of the form  $\phi(t) = \gamma$  where  $\gamma$  is a constant and  $f(t, \gamma) = 0$  for all  $t$ . In this case, the only singular solution is  $\phi(t) = 0$ .
- c. Rewrite the equation as

$$\frac{x'}{x} = 2 \sin t \cos t.$$

Integrate both sides with respect to  $t$ ,

$$\int \frac{x'}{x} dt = \int \frac{dx}{x} = \int 2 \sin t \cos t dt.$$

Note that there are several ways to integrate the right-hand side. For example,

$$\int 2 \sin t \cos t dt = \sin^2 t + c,$$

$$\int 2 \sin t \cos t dt = -\cos^2 t + c,$$

$$\int 2 \sin t \cos t dt = \int \sin 2t dt = -\frac{1}{2} \cos 2t + c.$$

This leads to different forms of the answer that can be obtained from each other through various trigonometric identities. Suppose we choose the first integral above. Then integration leads to

$$\ln |x| = \sin^2 t + c_1,$$

where  $c_1$  is a real constant. Take the exponential of both sides

$$|x| = c_2 e^{\sin^2 t}, \quad c_2 = e^{c_1} > 0.$$

Now we must consider the sign of  $x$ . When  $x > 0$ , we have

$$x = c_2 e^{\sin^2 t}, \quad c_2 > 0.$$

When  $x < 0$ , we have

$$x = -c_2 e^{\sin^2 t}, \quad c_2 > 0.$$

When  $x = 0$ , we have  $c_2 = 0$ . However, we know that  $c_2 > 0$ . Thus, we can not deal with this case here; we will include it via the singular solution. The analysis above allows us to drop the absolute value sign and rewrite the formula as

$$x = c_3 e^{\sin^2 t}, \quad c_3 \neq 0.$$

Finally, including the singular solution, we obtain the general solution

$$x(t) = c e^{\sin^2 t}.$$

where  $c$  is a real constant. If you used a different integral, then your answer will be in another form. However, you should be able to put it in this form by using a trigonometric identity

- d.** The initial condition implies that  $x(0) = c = 1$ . Thus, the particular solution associated with the initial condition is  $\psi(t) = e^{\sin^2 t}$ . If you used a different integral in Part (c), then your answer will be in another form. However, you should be able to put it in this form by using a trigonometric identity

2. Consider the ODE

$$x' = x \tan t + t \sec t.$$

- a. Solve the homogeneous part of the ODE.
- b. Solve the nonhomogeneous ODE using the method of variation of parameters.

### SOLUTION

a. The homogeneous part of the ODE is

$$X' = X \tan t,$$

where we have used  $X(t)$  to represent the solution of the homogeneous equation (Recall that we use  $x(t)$  to represent the solution of the nonhomogeneous equation.) This equation can be solved using separation of variables. First, note that the only singular solution is  $\phi(t) = 0$ . Next, rewrite the equation as

$$\frac{X'}{X} = \tan t.$$

Now, integrate both sides with respect to  $t$ . The right-hand side can be integrated using the substitution  $u = \cos t$ . The resulting equation is

$$\ln |X| = -\ln |\cos t| + c_1,$$

where  $c_1$  is a real constant. Take the exponential of both sides and obtain

$$|X| = \frac{c_2}{|\cos t|},$$

where  $c_2$  is a positive real constant. If we consider the four cases,

- i.  $x > 0$ , and  $\cos t > 0$ ,
- ii.  $x > 0$ , and  $\cos t < 0$ ,
- iii.  $x < 0$ , and  $\cos t > 0$ ,
- iv.  $x < 0$ , and  $\cos t < 0$ ,

then we find that we can rewrite the expression as

$$X = \frac{c_3}{\cos t},$$

where  $c_3$  is a real constant and  $c_3 \neq 0$ . Finally, including the singular solution  $\phi(t) = 0$ , we obtain the general solution of the homogeneous equation,

$$X(t) = \frac{c}{\cos t},$$

where  $c$  is a real constant.

- b. To solve the nonhomogeneous ODE using the method of variation of parameters, we assume a solution of the form

$$x(t) = \frac{c(t)}{\cos t},$$

where  $c$  is now a function of  $t$ . If we differentiate this solution using the product rule, we obtain

$$x'(t) = \frac{c'(t)}{\cos t} + \frac{c(t) \sin t}{\cos^2 t}.$$

Substitute  $x$  and  $x'$  into the nonhomogeneous equation and obtain

$$\frac{c'(t)}{\cos t} + \frac{c(t) \tan t}{\cos t} = \frac{c(t) \tan t}{\cos t} + \frac{t}{\cos t}.$$

The terms including  $c(t)$  cancel and we are left with

$$c'(t) = t.$$

This is readily integrated and we find that

$$c(t) = \frac{1}{2}t^2 + k,$$

where  $k$  is a real constant. Therefore, the general solution to the nonhomogeneous problem is

$$x(t) = \frac{\frac{1}{2}t^2 + k}{\cos t},$$

3. Consider the equation

$$xy' = x \cos^2 \frac{y}{x} + y.$$

- a. Use the substitution  $u = \frac{y}{x}$  to derive the ODE satisfied by  $u(x)$ .
- b. Determine the solution  $u(x)$  of the equation derived in Part (a).
- c. Determine the solution  $y(x)$  of the original equation.

**Note:** The following derivative may be useful:

$$\frac{d}{du} \tan u = \frac{1}{\cos^2 u}.$$

### SOLUTION

a. Rewrite the ODE in the normal form

$$y' = \cos^2 \frac{y}{x} + \frac{y}{x}.$$

The substitution  $u = \frac{y}{x}$  implies that

$$y' = u'x + u.$$

Therefore, applying the substitution leads to

$$u'x + u = \cos^2 u + u,$$

which can be rewritten as

$$\frac{u'}{\cos^2 u} = \frac{1}{x}.$$

b. Integrating with respect to  $x$  using the hint above, we get

$$\tan u = \ln |x| + c.$$

This is an implicit formula for  $u(x)$ . It is tempting to take the inverse tangent of both sides and obtain an explicit representation for  $u$ . If you do, you must recall that the range of  $\arctan(x)$  is  $(-\pi/2, \pi/2)$ . Therefore,  $u$  is limited to this range and this must be noted. Since we do not want to limit the values of our dependent variable, we choose to leave our answer in explicit form. Since there are no singular solutions, this is the general solution for  $u(x)$ .

c. To obtain the general solution for the original equation, we let  $u = \frac{y}{x}$  in the general solution given in Part (b). We find that

$$\tan \frac{y}{x} = \ln |x| + c.$$

No further simplification is possible.

4. Consider the following 1-parameter family of ODE's

$$y' = (c - y^3)(c - y).$$

- a. Sketch the bifurcation diagram in the  $c - y$  plane.
  - i. Identify all bifurcation points by their coordinate positions.
  - ii. Classify all branches of equilibrium points.
- b. From your bifurcation diagram sketch all the distinct phase-lines encountered as  $c$  increases.
- c. From the phase line at  $c = 1/2$  sketch the solution trajectories in the  $t - y$  plane.

### SOLUTION

- a. See the accompanying figures. Analysis:

Equilibrium branches in  $c$ - $y$  plane:

$$f(y; c) = (c - y^3)(c - y) \equiv 0 \quad \Rightarrow \quad y = c \quad \text{and} \quad y = c^{\frac{1}{3}}.$$

Bifurcation points are equilibrium points ( $f(y; c) = 0$ ) that are nodes i.e.,

$$\partial_y f(y; c) = -(c - y^3) - 3y^2(c - y) = 0$$

So, for  $y = c$

$$\partial_y f(c; c) = -(c - c^3) = 0$$

implies  $c = -1, 0, 1$  and bifurcation points  $(-1, -1), (0, 0), (1, 1)$ . For  $y = c^{\frac{1}{3}}$ ,

$$\partial_y f(c^{\frac{1}{3}}; c) = -3c^{\frac{2}{3}}(c - c^{\frac{1}{3}}) = 0$$

which again implies  $c = -1, 0, 1$  and bifurcation points  $(-1, -1), (0, 0), (1, 1)$ .

- b. and c. See the accompanying figures.

5. a. Determine whether the theorems of Peano and Cauchy apply for the following initial value problems. Motivate your answer by using the theorems.

i.  $x' = t^2x + \cos t, \quad x(\pi) = 1.$

ii.  $x' = \frac{x+t}{t}, \quad x(0) = 0.$

iii.  $x' = 4x^{\frac{3}{5}}, \quad x(0) = 0.$

- b. Consider the ODE

$$tx' = \frac{x^2 - t^2}{x}.$$

For which of the following initial conditions  $x(t_0) = x_0$  does the Cauchy theorem guarantee a unique solution of the ODE in some interval centered at  $t_0$ .

i.  $x(0) = 2.$

ii.  $x(0) = 0.$

iii.  $x(1) = 0.$

iv.  $x(1) = 2.$

v.  $x(35) = 35.$

### SOLUTION

Peano theorem indicates that if the vector field  $f(t, x)$  (for ODE  $x' = f(t, x)$ ) is continuous at the initial value  $x(t_0) = x_0$  then a solution exists on some interval  $(t_0 + \epsilon, t_0 + \epsilon)$ . In addition to this, if  $\partial/\partial_x f(t, x)$  is also continuous at the initial value then by Cauchy theorem the solution is also unique.

- a. i.  $f(t, x) = t^2x + \cos t$  is continuous for all  $t, x \rightarrow$  solution to IVP exists.  
 ii.  $f(t, x) = \frac{x+t}{t}$  is discontinuous at  $t = 0 \rightarrow$  continuous solution to IVP does not exist.  
 iii.  $f(t, x) = 4x^{\frac{3}{5}}$  is continuous for all  $t, x \rightarrow$  solution to IVP exists.
- b. The vector field  $f(t, x) = \frac{x^2 - t^2}{(tx)}$  and  $\partial/\partial_x f(t, x)$  are discontinuous at  $t = 0$  and  $x = 0$ . Therefore
- i.  $x(0) = 2 \rightarrow$  continuous solution to IVP does not exist (non-unique).  
 ii.  $x(0) = 0 \rightarrow$  continuous solution to IVP does not exist (non-unique).  
 iii.  $x(1) = 0 \rightarrow$  continuous solution to IVP does not exist (non-unique).  
 iv.  $x(1) = 2 \rightarrow$  continuous solution to IVP exists and is unique.  
 v.  $x(35) = 35 \rightarrow$  continuous solution to IVP exists and is unique.