On the front of your bluebook, write your name, instructor's name, and recitation number.

There are **NINE QUESTIONS**. You must **WORK EIGHT OUT OF NINE** problems. **CLEARLY IDENTIFY THE EIGHT PROBLEMS YOU WANT GRADED.** Start each problem on a new page, **SHOW ALL YOUR WORK** in your bluebook, and $\boxed{\text{box}}$ in your answers. Calculators, books, and crib sheets are **NOT** permitted.

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1. (25 points) Indicate whether the following statements are **TRUE** or **FALSE**. No justification is necessary.

a. (5 points) The equation $x' = x^2 - 1$ has an equilibrium at $x = 1$ that is a source.

b. (5 points) The initial-value problem $y' = y(1-y)^{1/3}$ with $y(t_0) = y_0$ has a unique solution for all (t_0, y_0).

c. (5 points) The system of equations

$$\begin{pmatrix} 1 & 5 & 15 & -2 \\ -2 & 8 & 1 & 4 \\ -3 & -9 & 2112 & 6 \\ 4 & 7 & 8 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ -7 \\ -13 \end{pmatrix}$$

has a unique solution.

d. (5 points) The vector functions

$$\mathbf{v}_1 = \begin{pmatrix} \mathrm{e}^t \\ \mathrm{e}^t \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^t \end{pmatrix}$$

are linearly dependent.

e. (5 points) Consider the homogeneous 2×2 system with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Suppose λ_1 and λ_2 are the eigenvalues of $\mathbf{A}$ and $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = -5$, then the equilibrium point $\mathbf{x} = \mathbf{0}$ must be a spiral sink.

SOLUTION

a. True.

b. False.

c. False.

d. False.

e. False.

2. (25 points) Solve the following differential equations:

   a. (10 points) $y' = 3e^{2\ln t + y}$

   b. (15 points) $y' + y = e^{-t}\sin t$

SOLUTION

a. The ODE is separable:

$$y' = 3t^2 e^y \Rightarrow e^{-y}y' = 3t^2.$$

Integrate with respect to t:

$$\begin{aligned} \int e^{-y}dy &= \int 3t^2 dt \\ -e^{-y} &= t^3 + C_1 \\ e^{-y} &= -t^3 + C_2 \\ e^y &= \frac{1}{C_2 - t^3} \\ y(t) &= \ln\left|\frac{1}{C_2 - t^3}\right| \end{aligned}$$

b. The ODE is linear. Thus, we can use either variation of parameters or an integrating factor.

Variation of Parameters:

$$\begin{aligned} \text{Homogeneous solution :} \quad & y_h(t) = Ce^{-t} \\ \text{Particular solution :} \quad & y_p(t) = C(t)e^{-t} \end{aligned}$$

Substitute the particular solution into the ODE and obtain

$$\begin{aligned} C' &= \sin t \\ C(t) &= -\cos t + K \end{aligned}$$

This gives particular solution $y_p(t) = (-\cos t + K)e^{-t}$. Ignoring or absorbing constant K gives the general solution

$$y(t) = C_1 e^{-t} - e^{-t}\cos t.$$

Integrating Factor:

$$\text{Integrating factor :} \quad e^t$$

$$\begin{aligned} e^t(y' + y) &= e^t(\sin t e^{-t}) \\ (e^t y)' &= \sin t \\ e^t y &= -\cos t + C \\ y(t) &= Ce^{-t} - e^{-t}\cos t. \end{aligned}$$

3. (25 points) Solve the following initial-value problem:

$$x'' = 2x^3, \qquad x(0) = 2, \qquad x'(0) = -4.$$

SOLUTION

The ODE $x'' = 2x^3$ has the form $x'' = f(x)$. Thus, multiply the ODE by x' and obtain

$$\begin{aligned} x'x'' &= x'(2x^3), \\ \left(\frac{1}{2}x'^2\right)' &= x'(2x^3). \end{aligned}$$

Integrate with respect to t:

$$\begin{aligned} \frac{1}{2}x'^2 &= \int (2x^3)dx, \\ \frac{1}{2}x'^2 &= \frac{1}{2}x^4 + C. \end{aligned}$$

Using the initial condition $x(0) = 2$ and $x'(0) = -4$, or equivalently, $(t_0, x_0, x_0') = (0, 2, -4)$, we get

$$\frac{1}{2}(-4)^2 = \frac{1}{2}(2)^4 + C.$$

and $C = 0$. Thus,

$$\begin{aligned} x'^2 &= x^4, \\ x' &= \pm x^2. \end{aligned}$$

Again the initial conditions imply

$$\begin{aligned} x' &= \pm x^2, \\ -4 &= \pm(2)^2. \end{aligned}$$

Thus, the minus sign is the appropriate choice and

$$x' = -x^2.$$

This equation is separable:

$$\begin{aligned} -x^{-2}x' &= 1, \\ -x^{-2}dx &= dt, \\ x^{-1} &= t + C_1, \\ x(t) &= \frac{1}{t + C_1}. \end{aligned}$$

Again the initial condition $x(0) = 2$ implies that $C_1 = 1/2$. Hence,

$$x(t) = \frac{1}{t + \frac{1}{2}}.$$

4. (25 points) Consider the following differential equation:

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 8x.$$

a. (5 points) Show that $y_1(x) = x$ and $y_2(x) = \frac{1}{x}$ form a fundamental set of solutions of the homogeneous equation.

b. (18 points) Use the method of variation of parameters to find a particular solution of the nonhomogeneous equation.

c. (2 points) Write down the general solution of the nonhomogeneous equation.

SOLUTION

a. First, we must show that $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous differential equation. The required derivatives are

$$y_1(x) = x, \qquad y_1'(x) = 1, \qquad y_1''(x) = 0,$$

$$y_2(x) = \frac{1}{x}, \qquad y_2'(x) = -\frac{1}{x^2}, \qquad y_2''(x) = \frac{2}{x^3}.$$

By substitution, we find that

$$y_1'' + \frac{y_1'}{x} - \frac{y_1}{x^2} = \frac{1}{x} - \frac{x}{x^2} = 0,$$

$$y_2'' + \frac{y_2'}{x} - \frac{y_2}{x^2} = \frac{2}{x^3} - \frac{1}{x^3} - \frac{1}{x^3} = 0.$$

Now, we must show that $y_1(x)$ and $y_2(x)$ are linearly independent. The Wronskian is

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = -\frac{2}{x} \neq 0.$$

Therefore, $y_1(x) = x$ and $y_2(x) = \frac{1}{x}$ are linearly independent and form a fundamental set of solutions.

b. The general solution of the homogeneous equation is

$$y(x) = C_1 x + \frac{C_2}{x}.$$

For the nonhomogeneous equation, we check a particular solution $\phi(x)$ of the form

$$\phi(x) = C_1(x)x + \frac{C_2(x)}{x}.$$

To find $C_1(x)$ and $C_2(x)$, we must solve the system of equations

$$C_1'(x)x + \frac{C_2'(x)}{x} = 0,$$

$$C_1'(x) - \frac{C_2'(x)}{x^2} = 8x.$$

The solution to this system is

$$C_1'(x) = 4x, \qquad C_2'(x) = -4x^3.$$

Therefore, we have

$$C_1(x) = \int 4x\,dx = 2x^2, \qquad C_2(x) = -\int 4x^3\,dx = -x^4.$$

Thus, the particular solution is

$$\phi(x) = (2x^2)x + (-x^4)\frac{1}{x} = x^3.$$

c. The general solution of the nonhomogeneous equation is

$$y(x) = C_1 x + \frac{C_2}{x} + \phi(x) = C_1 x + \frac{C_2}{x} + x^3.$$

5. (25 points) Consider the system of algebraic equations $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

a. (6 points) Compute the determinant of $\mathbf{A}$.

b. (4 points) What does your result in Part (a) imply about the existence of a solution of the system? What does it imply about the uniqueness of a solution?

c. (15 points) Use Gaussian elimination with back substitution to verify your answers in Part (b).

SOLUTION

a. The determinant of $\mathbf{A}$ is computed as follows:

$$\det \mathbf{A} = \begin{vmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 2(2-2) - 4(1-3) + 2(2-6) = 0.$$

b. Since $\det \mathbf{A} = 0$, the linear system has either no solution or an infinite number of solutions. This implies that if the solution exists, then it is not unique.

c. Apply the sequence of elementary row operations $\frac{1}{2}E_1 \to E_1$, $E_2 - E_1 \to E_2$, $E_3 - 3E_1 \to E_3$, $E_2 \leftrightarrow E_3$, and $-\frac{1}{2}E_2 \to E_2$ and obtain

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 1 & 2 & 1 & 3 \\ 3 & 2 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 2 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -2 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solve the resulting triangular linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_2 + x_3 &= 2 \end{aligned}$$

using back substitution and obtain

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - \frac{\alpha}{2} \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

where α is an arbitrary real constant. Therefore, the linear system has an infinite number of solutions.

6. (25 points) Consider the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

a. (15 points) Find the general solution of the system.

b. (10 points) Suppose λ is a double eigenvalue of the matrix $\mathbf{A}$ with corresponding eigenvector $\mathbf{u}$. Show that

$$\mathbf{x}(t) = \mathbf{v}\mathrm{e}^{\lambda t} + \mathbf{u}t\mathrm{e}^{\lambda t}$$

is a solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ only if $\mathbf{v}$ satisfies the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{u}.$$

SOLUTION

a. To find the general solution we must first determine the eigenvalues and eigenvectors of $\mathbf{A}$. The eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 = 0.$$

Therefore, $\lambda = \lambda_1 = \lambda_2 = 2$ is a double eigenvalue. Note that since $\mathbf{A}$ is a triangular matrix, the eigenvalues are simply the diagonal elements of $\mathbf{A}$. The eigenvectors corresponding to $\lambda = 2$ are solutions of the linear system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{u} = 0.$$

Any vector of form

$$\mathbf{u} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where α is an arbitrary constant is an eigenvector. We shall choose $\alpha = 1$. Since we have found only one eigenvector, we must solve the generalized eigenvalue problem $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{u}$ in order to obtain a second linearly independent solution. The resulting linear system is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The general solution of this linear system is

$$\mathbf{v} = \beta\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where β is an arbitrary constant. The first term is simply the eigenvector $\mathbf{u}$, therefore, we choose $\beta = 0$ to obtain a particular solution for $\mathbf{v}$. Finally, the general solution of the system of linear differential equations is

$$\mathbf{x}(t) = c_1\mathbf{u}\mathrm{e}^{\lambda t} + c_2\left[\mathbf{v}\mathrm{e}^{\lambda t} + \mathbf{u}t\mathrm{e}^{\lambda t}\right] = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{2t} + c_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}t\mathrm{e}^{2t}\right],$$

which can be rewritten as

$$x_1(t) = (c_1 + c_2 t)\mathrm{e}^{2t}, \qquad x_2(t) = c_2\mathrm{e}^{2t}.$$

b. Assume $\mathbf{x}$ is a solution of the linear system. Differentiate and obtain

$$\mathbf{x}' = \lambda \mathbf{v} e^{\lambda t} + \mathbf{u} e^{\lambda t} + \lambda \mathbf{u} t e^{\lambda t}.$$

Substitute $\mathbf{x}$ and $\mathbf{x}'$ into the linear system and obtain

$$\lambda \mathbf{v} e^{\lambda t} + \mathbf{u} e^{\lambda t} + \lambda \mathbf{u} t e^{\lambda t} = \mathbf{A}(\mathbf{v} e^{\lambda t} + \mathbf{u} t e^{\lambda t}) = \mathbf{A}\mathbf{v} e^{\lambda t} + \mathbf{A}\mathbf{u} t e^{\lambda t}.$$

Divide by $e^{\lambda t}$ and collect like powers of t to obtain

$$(\mathbf{A}\mathbf{v} - \lambda \mathbf{v} - \mathbf{u}) + (\mathbf{A}\mathbf{u} - \lambda \mathbf{u})t = \mathbf{0}.$$

Since λ and $\mathbf{u}$ are an eigenvalue-eigenvector pair, $\mathbf{A}\mathbf{u} - \lambda \mathbf{u} = \mathbf{0}$, and we get

$$\mathbf{A}\mathbf{v} - \lambda \mathbf{v} = (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{u}.$$

7. (25 points) Consider the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(t)$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} \mathrm{e}^t \\ 0 \end{pmatrix}.$$

a. (5 points) A complex-valued solution of the homogeneous system is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} \mathrm{e}^{2it}.$$

Find the real-valued general solution of the homogeneous system.

b. (15 points) Use the method of undetermined coefficients to find a particular solution of the nonhomogeneous system.

c. (5 points) Write down the general solution of the nonhomogeneous system.

SOLUTION

a. From the complex-valued solution, we know that $\lambda = 2i$ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector

$$\mathbf{u} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the real-valued general solution of the homogeneous system is

$$\mathbf{x}(t) = C_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right] + C_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t \right],$$

which can be simplified to

$$\mathbf{x}(t) = \begin{pmatrix} C_1 \cos 2t + C_2 \sin 2t \\ -C_1 \sin 2t + C_2 \cos 2t \end{pmatrix}.$$

b. We seek a particular solution $\phi_0(t) = (x_0(t), y_0(t))$ of the nonhomogeneous system of the form

$$x_0(t) = ae^t, \qquad y_0(t) = ce^t.$$

Substituting into the nonhomogeneous system, we obtain

$$a = 2c + 1, \qquad c = -2a.$$

The solution to this system of equations is

$$a = \frac{1}{5}, \qquad c = -\frac{2}{5}.$$

Therefore, the particular solution is

$$\phi_0(t) = \begin{pmatrix} \frac{1}{5} e^t \\ -\frac{2}{5} e^t \end{pmatrix}.$$

c. The general solution of the nonhomogeneous system is

$$\mathbf{x}(t) = \begin{pmatrix} C_1 \cos 2t + C_2 \sin 2t + \frac{1}{5}e^t \\ -C_1 \sin 2t + C_2 \cos 2t - \frac{2}{5}e^t \end{pmatrix}.$$

8. (25 points) Consider the following system of first-order differential equations describing the interaction of two species x and y as a function of time t:

$$\begin{aligned} x' &= x(3-x-2y) \\ y' &= y(2-y-x) \end{aligned}$$

a. (5 points) Find all the equilibrium points for the two species.

b. (20 points) Classify all the equilibrium points and state whether these points are structurally stable or structurally unstable.

SOLUTION

a. Equilibrium points occur for $\mathbf{x}' = (x', y') \equiv \mathbf{0}$. Thus, let

$$\begin{aligned} x' &= F_1(x,y) \quad \text{with} \quad F_1 := x(3-x-2y), \\ y' &= F_2(x,y) \quad \text{with} \quad F_2 := y(2-y-x), \end{aligned}$$

then

$$\begin{aligned} F_1 = 0 \quad &\Rightarrow \quad x = 0 \quad \text{or} \quad 3-x-2y = 0, \\ F_2 = 0 \quad &\Rightarrow \quad y = 0 \quad \text{or} \quad 2-y-x = 0. \end{aligned}$$

Selecting all possibilities gives

$$(i)\ (0,0), \quad (ii)\ (0,2), \quad (iii)\ (3,0), \quad (iv)\ (1,1).$$

c. To classify the equilibrium points, linearize about the equilibrium points $\mathbf{x_{eq}}$ with $\mathbf{x} = \mathbf{x_{eq}} + \mathbf{u}$ and where $\mathbf{u} = (u,v)^T$. Linearization implies $|\mathbf{u}| \ll 1$ and results in the linear system

$$\mathbf{u}' = \mathbf{A}\mathbf{u} \quad \text{with} \quad \mathbf{A} = \left.\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}\right|_{\mathbf{x}=\mathbf{x_{eq}}}.$$

Computing the partial derivatives, we obtain

$$\mathbf{A} = \left.\begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{pmatrix}\right|_{\mathbf{x}=\mathbf{x_{eq}}}.$$

Thus,

$$(i)\ (0,0) \Rightarrow \mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$. Therefore, $(0,0)$ is a source node and is structurally stable.

$$(ii)\ (0,2) \Rightarrow \mathbf{A} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix},$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore, $(0, 2)$ is a sink node and is structurally stable.

$$(iii)\ (3, 0) \Rightarrow \mathbf{A} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix},$$

with eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. Therefore, $(3, 0)$ is a sink node and is structurally stable.

$$(iv)\ (1, 1) \Rightarrow \mathbf{A} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix},$$

with eigenvalues $\lambda_1 = -1 + \sqrt{2}$ and $\lambda_2 = -1 - \sqrt{2}$. Therefore, $(1, 1)$ is a saddle node and is structurally stable.

9. (25 points) Consider the differential equation

$$x' + 2tx = 0.$$

a. (20 points) Find the first five terms (up to and including t^4) of the power series solution near the ordinary point $t = 0$.

b. (5 points) Find the general term and determine the complete power series solution near the ordinary point $t = 0$.

SOLUTION

a. Assume a power series solution of the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Differentiate and obtain

$$x'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}.$$

Substitute x and x' into the differential equation and obtain

$$\sum_{n=1}^{\infty} n a_n t^{n-1} + 2t \sum_{n=0}^{\infty} a_n t^n = 0,$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} 2 a_n t^{n+1} = 0.$$

Let $k = n - 1$ in the first series and $k = n + 1$ in the second series and obtain

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} t^k + \sum_{k=1}^{\infty} 2 a_{k-1} t^k = 0.$$

Rewrite the series as

$$a_1 + \sum_{k=1}^{\infty} [(k+1) a_{k+1} t^k + 2 a_{k-1}] t^k = 0.$$

Set the coefficients of t^k, $k = 0, 1, 2, \ldots$, equal to zero and find

$$a_1 = 0, \qquad a_{k+1} = -\frac{2a_{k-1}}{k+1}, \quad k \geq 2.$$

The first five coefficients are

$$a_0 \text{ arbitrary}, \qquad a_1 = 0, \qquad a_2 = -a_0, \qquad a_3 = 0, \qquad a_4 = \frac{a_0}{2}.$$

Thus, the (incomplete) power series solution is

$$x(t) = a_0 - a_0 t^2 + a_0 \frac{t^4}{2} + \cdots.$$

b. Calculate the next two (or more if necessary) coefficients and find

$$a_5 = 0, \qquad a_6 = -\frac{a_0}{3!}.$$

The general even and odd terms are

$$a_{2n} = \frac{a_0(-1)^n}{n!}, \qquad a_{2n+1} = 0,$$

for $n = 0, 1, 2, \ldots$, respectively. Thus, the complete power series solution near the ordinary point $t = 0$ is

$$x(t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}.$$

You are not required to sum the series, but note that

$$x(t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} = a_0 \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = a_0 e^{-t^2}.$$

This solution can be obtained by applying separation of variables to the differential equation.