Exam 1 Spring 2003 Solutions APPM 2360

1. (20pts) Classify each of the following differential equations: by (i) order, (ii) linear or nonlinear, (iii) separable or not, and *if* it is linear (iv) homogeneous or nonhomogeneous.

a) $y'' + ty' = y$ b) $y' + \frac{1}{t}y = y$ c) $y' = e^y - 1$ d) $yy'' + y = \sin(t)$

Answers:

a) is 2nd order, linear, nonseparable and homogeneous.
b) is 1st order, linear, separable, homogeneous
c) is 1st order, nonlinear, separable
d) is 2nd order, nonlinear, nonseparable

2. (20pts) Mark each of the following statements as True or False. (No work need be shown)

a) (4pts) Let $y_1(t)$ and $y_2(t)$ be solutions of the equation $y'' + t^3y' + y = 0$. Then $y(t) = y_1(t) - y_2(t)$ is also a solution.

b) (4pts) Let $y_1(t)$ be a solution of $y' - y + 1 = 0$. Then $2y_1(t)$ is also a solution.

c) (4pts) Picard's theorem implies that the solution of the initial value problem $y' + y^{1/3} = t, \quad y(0) = 0$ exists and is unique.

d) (4pts) Picard's theorem implies that the solution of the initial value problem $y' = \frac{12}{y} + \ln(t), \quad y(1) = 1$ exists and is unique.

e) (2pts) The function $y(t) = te^{-\frac{t^2}{2}}$ is a solution of the equation $y' + ty = 0$.

f) (2pts) The function $y(t) = 2t$ is the solution of the IVP $y' = \sin(ty - 2t^2) + 2, \quad y(1) = \frac{1}{2}$.

Answers

a) T. The is the superposition principle
b) F. The equation is nonhomogeneous, so superposition doesn't work
c) F. Picard's theorem doesn't apply at $y(0) = 0$.
d) T. Picard's theorem does apply since $f(t,y)$ and $\partial f/\partial y$ are continuous near (1,1)
e) F. Just substitute and find it doesn't solve the ode
f) F Solves the ode, but doesn't satisfy the I.C.

3. (20pts) Find the general solution of $ty' - t^3 y = t^3$ using the integrating factor method.

Answers

In standard form the equation is

$$y' - t^2 y = t^2$$

So (2 pts) $p(t) = -t^2$. The integrating factor is (6 pts) $\mu = e^{-\frac{t^3}{3}}$, and we obtain

$$\frac{d}{dt}\left[e^{-\frac{t^3}{3}} y\right] = e^{-\frac{t^3}{3}} t^2 \text{ (6 pts)}$$

Integrating both sides and solving gives (6 pts)

$$y = ce^{\frac{t^3}{3}} - 1$$

4. (20pts)

a) The temperature of a corpse left outside on a cold winter's day obeys the equation

$$\frac{dT}{dt} = -kT, \quad T(0) = 99°F$$

for some positive constant k. Find $T(t)$.

b) Suppose the corpse is instead left in a sauna. The temperature now obeys

$$\frac{dT}{dt} = k(120 - T), \quad T(0) = 99°F$$

where the temperature of the room is 120°. Find $T(t)$ using the method of variation of parameters.

c) For each part (a) and (b), what is the temperature of the corpse after a long time?

Answers

a) (6 pts)The general solution is $T(t) = ce^{-kt}$, putting in the initial condition yields $T(t) = 99e^{-kt}$

b) (10 pts) The homogeneous solution is the same as (a). Allowing the parameter *c* to vary gives $T_p(t) = v(t)e^{-kt}$ and substituting yields

$$\frac{d}{dt}\left(v(t)e^{-kt}\right) + kv(t)e^{-kt} = 120k \quad \Rightarrow \quad v' = 120ke^{kt}$$

so $v(t) = 120e^{kt}$, and $T(t) = T_h + T_p = ce^{-kt} + 120$. Putting in the initial condition yields

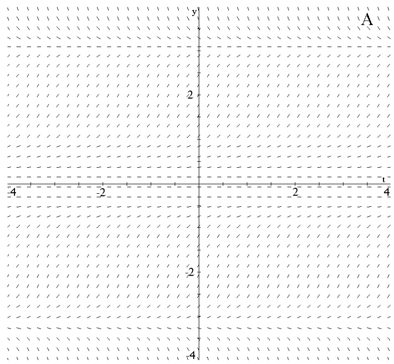
$$T(t) = (99 - 120)e^{-kt} + 120 = -21e^{-kt} + 120$$

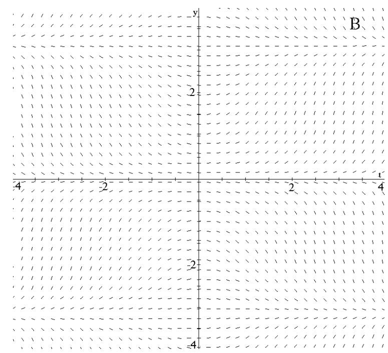
c) (4 pts) For a) $T(t) \to 0$, and for (b) $T(t) \to 120$. In both cases, the temperature approaches the ambient one.

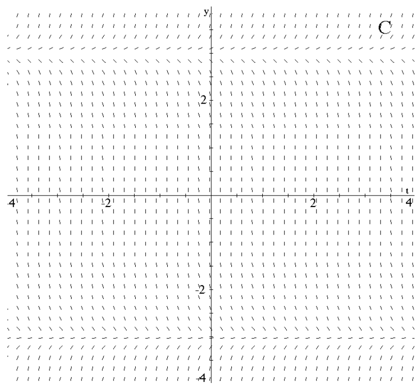
5. (20pts)

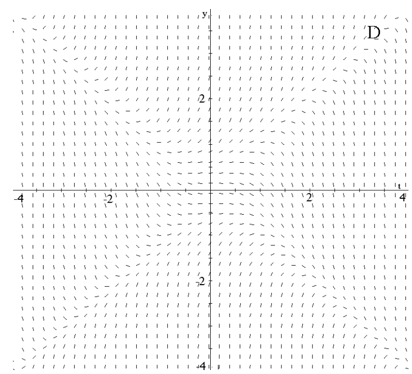
a) Match each of the following differential equations (1)-(4) with one of the direction fields (A)-(D).

(1) $\dfrac{dy}{dt} = y^2 - t^2$ (2) $\dfrac{dy}{dt} = t\sin(y)$ (3) $\dfrac{dy}{dt} = y^2 - 9$ (4) $\dfrac{dy}{dt} = y\sin(y)$









b) For equation (4), suppose $y(0) = 2$. What will be $\lim_{t \to \infty} y(t)$ and $\lim_{t \to -\infty} y(t)$?

c) For equation (3), what are the equilibria?

d) Classify the equilibria of (3) as stable, unstable or neither.

Answers

a) (12 pts)

(1) is nonautonomous, so it must be either B or D. Note that the zero isoclines for (1) are on the lines *y*= ± *t*. This is what is shown in D.

(2) is also nonautonomous. It has an equilibrium at *y* = 0, which is what B shows.

(3) is autonomous, so it must be A or C. It has equilibria at ±3, but not at 0. So it must be C.

(4) is autonomous and has equilibria at 0, and ±π. This is A.

b) (4 pts) Since *y*(0) = 2 we are between the equilibria at 0 and π. *f*(*y*) > 0, so the solution grows monotonically. Thus $\lim_{t \to \infty} y(t) = \pi$ $\lim_{t \to -\infty} y(t) = 0$

c) (2 pts) The equilibria are at $y = \pm 3$

d) (2 pts) 3 is unstable and –3 is stable.

Extra Credit (5pts). Solve the differential equation $\frac{dy}{dt} + ty = y^3$.

Answer:

This is an example of Bernoulli's equation. It can be linearized with the substitution

$$v = y^{1-3} = \frac{1}{y^2}$$

Substituting we obtain

$$\frac{dv}{dt} = -2\frac{1}{y^3}\frac{dy}{dt} = -2\frac{1}{y^3}\left(-ty + y^3\right) = 2tv - 2$$

This is linear but nonhomogeneous. The integrating factor is $\mu = e^{-t^2}$, and we obtain

$$\frac{d}{dt}(e^{-t^2}v) = -2e^{-t^2} \quad \Rightarrow \quad v = e^{t^2}\left(-2\int_0^t e^{-s^2}\,ds + c\right)$$

which gives

$$y = \pm\frac{e^{-\frac{1}{2}t^2}}{\left(c - 2\int_0^t e^{-s^2}\,ds\right)^{1/2}}$$

where the ± sign is determined by initial conditions.