INSTRUCTIONS:

- ON THE FRONT OF YOUR BLUEBOOK write your (1) name, (2) student ID number, (3) lecture section/instrcutor, and (4) a grading grid.
- Work all five problems. Start each problem on a new page.
- **Show your work** and clearly identify/box your final answer. A correct answer without the relevant work will receive no credit.
- A double-sided page of notes is permitted.
- Computers, calculators, and books are not permitted.

1. (20 points) Consider the system of ODE's

$$\begin{aligned} \frac{dx}{dt} &= 3x + 2y - 2 \\ \frac{dy}{dt} &= 2x - y + 1 \end{aligned}$$

   (a) (5 points) Plot the nullclines for this system on the (x, y)-phase plane, and plot the direction field along each nullcline.

   (b) (5 points) Rewrite the system as a matrix equation

$$\frac{d\mathbf{z}}{dt} = A\mathbf{z} - \mathbf{b}$$

   where $\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}$, A is a 2×2 matrix, and $\mathbf{b}$ is a 2 dimensional vector. What are A and $\mathbf{b}$?

   (c) (5 points) Solve for the equilibrium points by solving the algebraic system $A\mathbf{z}_0 - \mathbf{b} = 0$ for $\mathbf{z}_0$. You *must* solve this system using A^{-1}.

   (d) (5 points) Indicate on your phase portrait from (a) the equilibria that you found in (c) with circles and determine whether these equilibria are stable or unstable.

2. (20 points) Consider the following system of linear equations:

$$\begin{aligned} 2x + y + z - 2 &= 0 \\ -2y + x - 2z - 1 &= 0 \\ -4z + 3x - y + 3 &= 0\ . \end{aligned}$$

   (a) (3 points) Is this system homogeneous or nonhomogeneous?

   (b) (5 points) Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Write the above system in the form $A\mathbf{x} = \mathbf{b}$, and find A and $\mathbf{b}$.

   (c) (12 points) Solve the above system sing the *Reduced Row Echelon Form (Gauss-Jordan Elimination)*.

3. (20 points) Answer the following true/false and multiple-choice questions.

   (a) (4 points) The following system of equations

$$\begin{aligned} x + y + z &= 3 \\ -2x + y &= 1 \\ 3y + 2z &= 3 \end{aligned}$$

   has:

   (A) no solution

   (B) one solution

   (C) an infinite number of solutions.

   (b) (4 points) The set of vectors $\{[-3,-6],[1,0],[2,3]\}$ spans $\mathbb{R}^2$.

   (c) (4 points) The set of functions $\{\sin(t), t\sin(t)\}$ is linearly *independent* on the interval $t \in (-\infty, \infty)$.

   (d) (4 points) The matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ *commute.*

   (e) (4 points) The set of the solutions of the following system forms a vector space:

$$\begin{aligned} y &= 2x - z\ , \\ z - 1 &= x + y\ . \end{aligned}$$

4. (20 points)

   (a) (5 points) Can the vectors $u = [1, 0, -3]$ and $v = [2, 0, -6]$ be added? If so, find $2u - 3v$.

   (b) (5 points) Is the matrix

$$A = \begin{bmatrix} -3 & 0 \\ -6 & 1 \end{bmatrix}$$

   invertible? If so, what is $3A - I$?

   (c) (5 points) Consider the following matrices:

$$A = \begin{bmatrix} -3 & 0 & 1 \\ -6 & 1 & 0 \end{bmatrix}\ , \quad B = \begin{bmatrix} 0 & 1 \\ -6 & 1 \\ 2 & 2 \end{bmatrix}\ .$$

   Does the product BA exist? If so, find this product. Does the product AB exist? If so, find this product.

   (d) (5 points) Using determinants only, determine whether the set of vectors $\{[1,-1,0],[2,1,-1],[-3,0,1]\}$ forms a basis for $\mathbb{R}^3$.

5. (20 points) Solve the differential equation

$$y'' + y' - 12y = 0$$

with the initial values $y(0) = 0$, $y'(0) = 1$.