

1. (20 points) Consider the system of ODE's

$$\begin{aligned}\frac{dx}{dt} &= 3x + 2y - 2 \\ \frac{dy}{dt} &= 2x - y + 1\end{aligned}$$

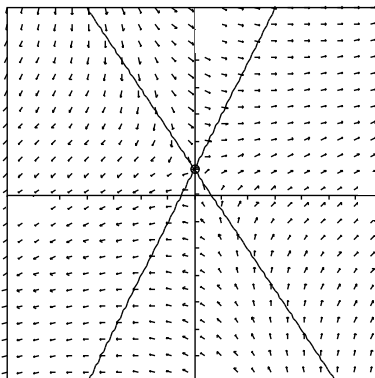
- (a) (5 points) Plot the nullclines for this system on the (x, y) -phase plane, and plot the direction field along each nullcline.
 (b) (5 points) Rewrite the system as a matrix equation

$$\frac{d\mathbf{z}}{dt} = A\mathbf{z} - \mathbf{b}$$

where $\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}$, A is a 2×2 matrix, and \mathbf{b} is a 2 dimensional vector. What are A and \mathbf{b} ?

- (c) (5 points) Solve for the equilibrium points by solving the algebraic system $A\mathbf{z}_0 - \mathbf{b} = 0$ for \mathbf{z}_0 . You *must* solve this system using A^{-1} .
 (d) (5 points) Indicate on your phase portrait from (a) the equilibria that you found in (c) with circles and determine whether these equilibria are stable or unstable.

SOLUTION



- (a) The nullclines meet at the (equilibrium) point $(x, y) = (0, 1)$.

(b) $A = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(c) $A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$ and $\mathbf{z}_0 = A^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- (d) Unstable ("saddle" type).

2. (20 points) Consider the following system of linear equations:

$$\begin{aligned} 2x + y + z - 2 &= 0 \\ -2y + x - 2z - 1 &= 0 \\ -4z + 3x - y + 3 &= 0. \end{aligned}$$

(a) (3 points) Is this system homogeneous or nonhomogeneous?

(b) (5 points) Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Write the above system in the form $A\mathbf{x} = \mathbf{b}$, and find A and

\mathbf{b} .

(c) (12 points) Solve the above system using the *Reduced Row Echelon Form (Gauss-Jordan Elimination)*.

SOLUTION

(a) Nonhomogeneous.

(b) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -2 \\ 3 & -1 & -4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$.

(c) $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & -2 & -2 & 1 \\ 3 & -1 & -4 & -3 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & -4 & -3 \end{array} \right]$

$$\xrightarrow{R_2 \rightarrow (R_2 - 2R_1)/5} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & -1 & -4 & -3 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 2 & -6 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & -6 \end{array} \right].$$

To this end, we find that $-3z = -6 \implies z = 2$; $y + z = 0 \implies y = -2$; and finally

$$x - 2y - 2z = 1 \implies x = 1. \text{ Therefore } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

3. (20 points) Answer the following true/false and multiple-choice questions.

(a) (4 points) The following system of equations

$$\begin{aligned}x + y + z &= 3 \\ -2x + y &= 1 \\ 3y + 2z &= 3\end{aligned}$$

has:

- (i) no solution
 - (ii) one solution
 - (iii) an infinite number of solutions.
- (b) (4 points) The set of vectors $\{-3, -6\}, [1, 0], [2, 3]\}$ spans \mathbb{R}^2 .
- (c) (4 points) The set of functions $\{\sin(t), t \sin(t)\}$ is linearly *independent* on the interval $t \in (-\infty, \infty)$.
- (d) (4 points) The matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ *commute*.
- (e) (4 points) The set of the solutions of the following system forms a vector space:

$$\begin{aligned}y &= 2x - z, \\ z - 1 &= x + y.\end{aligned}$$

SOLUTION

- (a) **(i)** as $R_2 \rightarrow R_2 + 2R_1$ shows that R_2 and R_3 are inconsistent
- (b) **True** as the first two vectors are linearly independent and therefore span \mathbb{R}^2
- (c) **True** as $|W| \neq 0$
- (d) **True** as $AB = BA = \begin{pmatrix} 4 & 8 \\ 8 & 16 \end{pmatrix}$.
- (e) **False** as the system is nonhomogeneous

4. (20 points)

(a) (5 points) Can the vectors $u = [1, 0, -3]$ and $v = [2, 0, -6]$ be added? If so, find $2u - 3v$.

(b) (5 points) Is the matrix

$$A = \begin{bmatrix} -3 & 0 \\ -6 & 1 \end{bmatrix}$$

invertible? Also, calculate $3A - I$.

(c) (5 points) Consider the following matrices:

$$A = \begin{bmatrix} -3 & 0 & 1 \\ -6 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -6 & 1 \\ 2 & 2 \end{bmatrix}.$$

Does the product BA exist? If so, find this product. Does the product AB exist? If so, find this product.

(d) (5 points) Using determinants only, determine whether the set of vectors $\{[1, -1, 0], [2, 1, -1], [-3, 0, 1]\}$ forms a basis for \mathbb{R}^3 .

SOLUTION

(a) $2u - 3v = [-4, 0, 12]$.

(b) A is invertible as $\det(A) \neq 0$. In fact, $A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 \\ -2 & 1 \end{bmatrix}$.

In addition, $3A - I = \begin{bmatrix} -10 & 0 \\ -18 & 2 \end{bmatrix}$.

(c)

$$BA = \begin{bmatrix} -6 & 1 & 0 \\ 12 & 1 & -6 \\ -18 & 2 & 2 \end{bmatrix}.$$

and

$$AB = \begin{bmatrix} 2 & -1 \\ -6 & -5 \end{bmatrix}.$$

(d) The determinant is zero, hence they do not form a basis.

5. (20 points) Solve the differential equation

$$y'' + y' - 12y = 0$$

with the initial values $y(0) = 0$, $y'(0) = 1$.

SOLUTION

- (a) (7 points) The characteristic equation is

$$r^2 + r - 12 = (r - 3)(r + 4) = 0 .$$

Its roots are $r_1 = 3$ and $r_2 = -4$.

- (b) (6 points) The general solution of this homogeneous equation is

$$y = c_1 e^{3t} + c_2 e^{-4t} .$$

- (c) (6 points) The initial conditions require that

$$y(0) = c_1 + c_2 = 0 , \quad y'(0) = 3c_1 - 4c_2 = 1 .$$

- (d) (6 points) Solving gives that $c_1 = 1/7$, $c_2 = -1/7$, and

$$y(t) = \frac{e^{3t} - e^{-4t}}{7} .$$