You must work all of the problems on the exam. Show **ALL** of your work in your bluebook, and **box** in your final answers. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Start each problem on the **top of a new page**. You may have one sheet of notes (8 1/2 by 11 inches). No other notes, books, calculators, nuclear weapons, etc. are allowed.

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1. (25 points) Find the explicit solution to the initial-value problem

$$y' = ty^2 + ty \ , \qquad y(0) = 1 \ .$$

2. (25 points) Consider the following nonlinear system of ODE's:

$$\begin{aligned} x' &= x(2 - x - y) \ , \\ y' &= y(y - x) \ . \end{aligned}$$

   (a) Find and plot all of the nullclines.

   (b) Find and plot all of the equilibrium points on the same graph as (a).

   (c) Classify the equilibria points as repelling node, saddle, etc. Also give their stability.

3. (25 points) Consider the initial-value problem $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 5 & 2 \\ -2 & 9 \end{pmatrix} \ , \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ .$$

   (a) Find the general solution of the homogeneous problem.

   (b) Solve the initial-value problem.

4. (25 points) For each of the following equations write down the *form* of the *particular* solution according to the method of **undetermined coefficients** (you do NOT need to find the values of the coefficients):

   (a) $y'' + 6y' + 9y = t(1 - t)$

   (b) $y'' + 6y' + 9y = e^{-3t}$

   (c) $y'' + 4y = \sin(t)$

   (d) $y'' = t(1 - t)$.

5. (25 points) Consider the following ODE:

$$y' + p(t)y = e^t \cos t \ .$$

   (a) Find $p(t)$ if the homogeneous solution is given by $y_h = c\cos t$.

   (b) Find the general solution to the above ODE. (Hint: **Note:** you can do part (b) without part (a). You might use the integral $\int \tan(x)dx = -\ln|\cos x| + c$.)

6. (25 points) Mark the following as **True** or **False**. No work need be shown.

   (a) The functions $\sin(t)$ and $\sin(t+\pi/2)$ are linearly dependent.

   (b) The initial-value problem $\dot{x} = 1 + tx^{1/7}$ with $x(1) = 0$ is guaranteed to have a unique solution by Picard's Theorem of existence and uniqueness.

   (c) The matrix

$$\begin{pmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

   is invertible.

   (d) The set $V = \text{Span}\{1, x^2, 1 + x^2\}$ is a vector space.

   (e) Let $y_1 = e^t$ and $y_2 = e^{-2t}$ be solutions of $y'' + ay' + by = 0$. Then $a + b = -1$.

   (f) The vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ 1 & 2 & -2 \end{pmatrix}$.

7. (25 points) Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -9 & 3 & 3 \\ -1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix}$.

   (a) Does $AB = BA$? Why or why not?

   (b) Find B^{-1} if it exists.

   (c) Compute the determiniant of A.

   (d) Find a value of k for which the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution. (You do not need to solve the system for $\mathbf{x}$).

8. (25 points) No work need be shown for this problem.

   a) Match each of the four systems of differential equations (1)-(4) below, to the corresponding phase plane (A)-(D).

$$(1)\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2\pi x) \end{pmatrix}, \qquad (2)\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y(1-x) \\ -x(1-4x^2) \end{pmatrix}$$

$$(3)\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -x + (1-8x^2)y \end{pmatrix}, \qquad (4)\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sin(2\pi y) \\ y - \sin(2\pi x) \end{pmatrix}$$

   b) Classify the equilibria of system (2).

   c) Which figure contains a limit cycle?

   d) For system (1), are the equations autonomous or nonautonomous? linear or nonlinear?