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3. Consider the initial-value problem $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 5 & 2 \\ -2 & 9 \end{pmatrix}$$

and $\mathbf{x}(0) = [0, -1]^T$.

(a) Find the general solution of the homogeneous problem.

(b) Solve the initial-value problem.

Solution

(a) (5 pts) The characteristic equation is

$$\lambda^2 - 14\lambda + 49 = (\lambda - 7)^2 = 0 \ ,$$

whose eigenvalues are $\lambda_1 = \lambda_2 = 7$.

(b) (5 pts) The augmented form for finding the first (and only) eigenvector is

$$[A - 7I \mid 0] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -2 & 2 & 0 \end{array}\right] \ ,$$

from which the eigenvector (up to c) is $\mathbf{v_1} = [1, 1]$.

(c) (5 pts) The augmented form for finding the generalized eigenvector is

$$[A - 7I \mid \mathbf{u}] = \left[\begin{array}{cc|c} -2 & 2 & 1 \\ -2 & 2 & 1 \end{array}\right] \ ,$$

from which the generalized eigenvector is $\mathbf{u} = [-\frac{1}{2}, 0]$.

(d) (5 pts) The general solution of the homogeneous problem is given by

$$\mathbf{x} = c_1 e^{7t}\mathbf{v_1} + c_2 e^{7t}(t\mathbf{v_1} + \mathbf{u}) = \begin{pmatrix} c_1 + c_2(t - \frac{1}{2}) \\ c_1 + c_2 t \end{pmatrix} \ .$$

(e) (5 pts) With the given i.c. the constants are found to be $c_1 = -1, c_2 = -2$ and the solution of the IVP is

$$\mathbf{x} = -\begin{pmatrix} 2t \\ 1 + 2t \end{pmatrix} \ e^{7t}.$$

4 For each of the following equations write down the form of the particular solution according to the method of **undetermined coefficients** (do NOT find the parciular solution):

(a) $y'' + 6y' + 9y = t(1 - t)$

(b) $y'' + 6y' + 9y = e^{-3t}$

(c) $y'' + 4y = \sin(t)$

(d) $y'' = t(1 - t)$.

Solution

NOTE: No points are taken off if the homogeneous solution is included. 6 pts each with one point for free.

(a) The eigenvalues are $r_1 = r_2 = -3$ and the basis functions are $\{e^{-3t}, te^{-3t}\}$. Therefore, $f(t)$ is not spanned by the basis and $y_p = At^2 + Bt + C$.

(b) Same as first one, but now $f(t)$ is spanned by the basis and $y_p = At^2e^{-3t}$.

(c) The eigenvalues are $r_{1,2} = \pm 2i$ and the basis functions are $\{\cos(2t), \sin(2t)\}$. The guess is thus $y_p = A\cos(t) + B\sin(t)$.

(d) The eigenvalues are zero, the basis is $\{1, t\}$, and part of $f(t)$ is spanned by the basis. The minimal guess is $y_p = At^4 + Bt^3$. Another way to answer/verify is simply to integrate twice (note that the free constant is spanned by the homogeneous soltution).

6

1. (25 points) Mark the following as **True** or **False**. No work need be shown.

   (a) The functions $\sin(t)$ and $\sin(t + \pi/2)$ are linearly dependent.

   (b) The initial-value problem $\dot{x} = 1 + tx^{1/7}$ with $x(1) = 0$ is guaranteed to have a unique solution by Picard's Theorem of existence and uniqueness.

   (c) The matrix

$$\begin{pmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

   is invertible.

   (d) The set $V = \text{Span}\{1, x^2, 1 + x^2\}$ is a vector space.

   (e) Let $y_1 = e^t$ and $y_2 = e^{-2t}$ be solutions of $y'' + ay' + by = 0$. Then $a + b = -1$.

   (f) The vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ 1 & 2 & -2 \end{pmatrix}$.

SOLUTION

(a) False. $\sin(t + \pi/2) = \cos(t)$, and $c_1 \sin(t) + c_2 \cos(t) = 0$ for all t only when $c_1 = c_2 = 0$.

(b) False. $\partial f/\partial y$ is infinite at x = 0, so the hypothesis needed for uniqueness fails.

(c) True. The determinant of an upper triangular matrix is the product of its diagonal elements. In this case, -4, which is nonzero, so the inverse exists.

(d) True. The span of *any* set of vectors is a vector space.

(e) False. Just compute $A\mathbf{x} = [2, -2, 1]^T$. But this is not proportional to $\mathbf{x}$ so it can't be $\lambda\mathbf{x}$.

7 Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -9 & 3 & 3 \\ -1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix}$.

(a) Does $AB = BA$? Why or why not?

(b) Find B^{-1} if it exists.

(c) Compute the determiniant of A.

(d) Find a value of k for which the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution. (You do not need to solve the system for $\mathbf{x}$).

SOLUTION

(a) (4 pts) No. In fact neither of the products exist.

(b) (6 pts) $B^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

(c) (6 pts) $|A| = -1 \begin{vmatrix} -1 & 1 \\ 3 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ -9 & 3 \end{vmatrix} = -1(-3-3) + (3-9) = 0.$

(d) (9 pts) When you row reduce the augmented matrix $A|b$ using $9R_1 + R_2 \to R_2$, $-R_1 + R_2 \to R_3$, $R_2/6 \to R_2$, $-R_2 + R_3 \to R_3$ you get $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & k-1 \end{pmatrix}$. Thus $k = 1$.

8 a) Match each of the four systems of differential equations (1)-(4) below, to the corresponding phase plane (A)-(D).

$$(1) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2\pi x) \end{pmatrix}, \qquad (2) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y(1-x) \\ -x(1-4x^2) \end{pmatrix}$$

$$(3) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -x + (1-8x^2)y \end{pmatrix}, \qquad (4) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sin(2\pi y) \\ y - \sin(2\pi x) \end{pmatrix}$$

b) Classify the equilibria in Figure (B).

c) Which figure contains a limit cycle?

d) For system (1), are the equations autonomous or nonautonomous? linear or nonlinear?

SOLUTION

1. (12 pts) 1 is C, 2 is B, 3 is A and 4 is D
2. (5 pts) There are two saddles at $(\pm\frac{1}{2}, 0)$ and a center at $(0, 0)$.
3. (4 pts) A
4. (4 pts) autonomous, nonlinear.