Exam 2 Solutions

1. Consider the following first-order system of ODE's

$$\begin{aligned}\dot{x} &= 2x - xy\\ \dot{y} &= -3y + \frac{1}{2}xy\end{aligned}$$

a. Find all the equilibrium points (5 points)

b. Find and sketch the x-nullcline(s). Annotate the nullcline with the appropriate vector directions. (5 points)

c. Find and sketch the y-nullcline(s). Annotate the nullcline with the appropriate vector directions. (5 points)

d. Sketch the complete phase plane by combining your answers from a.–c. In the open regions of the phase plane sketch the vector directions. (5 points)

SOLUTION

a. The only possibilities for equilibrium points are $(0,0)$ and $(6,2)$ this follows from

$$\begin{aligned}0 &= x(2-y)\\ 0 &= y(-3+\frac{1}{2}x)\end{aligned}$$

b. x-nullclines are the lines $x = 0$ and $y = 2$.
On $x = 0$ we have (i) $\downarrow$ for $y > 0$ and (ii) $\uparrow$ for $y < 0$.
On $y = 2$ we have (i) $\downarrow$ for $x < 2$ and (ii) $\uparrow$ for $x > 2$.

c. y-nullclines are the lines $y = 0$ and $x = 6$.
On $y = 0$ we have (i) $\leftarrow$ for $x < 0$ and (ii) $\rightarrow$ for $x > 0$.
On $x = 6$ we have (i) $\rightarrow$ for $y < 2$ and (ii) $\leftarrow$ for $y > 2$.

d. put together.

2. Solve the following the system of equations by Gaussian Elimination or putting it into Row Reduced Echelon Form (20 points).

$$\begin{aligned} x_1 + x_2 + x_4 &= 2\ , \\ 2x_1 + x_2 - x_3 + x_4 &= 1\ , \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4\ , \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3\ . \end{aligned}$$

SOLUTION

in Augmented form we have

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 4 & 1 & 1 & 9 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & -3 & -3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim$$

3. Consider the system of equations

$$\begin{aligned} -x - 3y + z &= 0\,, \\ 9x + 5y + 2z &= 0\,, \\ 2y - z &= 0\,. \end{aligned}$$

a. Do these equations have a unique solution? (6 points)

b. Find the span of the set of solutions of these equations. (10 points)

c. Does the set of solutions of these equations form a basis for $\mathbb{R}^3$? (4 points)

SOLUTION

a. No. This can be established by showing that their determinant is zero:

$$\begin{vmatrix} -1 & -3 & 1 \\ 9 & 5 & 2 \\ 0 & 2 & -1 \end{vmatrix} \overset{3^{rd}\text{ row}}{=} 2(-2-9) + 1(-5+27) = -22 + 22 = 0\,.$$

Another way to establish this is from the RREF (see part c).

b. First find the RREF

$$\begin{bmatrix} -1 & -3 & 1 \\ 9 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -1 \\ 9 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -1 \\ 0 & -22 & 11 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -1 \\ 0 & -22 & 11 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

The RREF shows that the vectors span those coordinates (x, y, z) that satisfy the equations $x + z/2 = 0$ and $y - z/2 = 0$. Hence, the span is a line (one degree of freedom) in $\mathbb{R}^3$ that can be parametrized by $(-1, 1, 2)t$.

c. No, they cannot be a basis of $\mathbb{R}^3$ because they span only a line.

4. **a.** Is the set $\{1, t, 2t^2 - 1, 4t^3 - 3t\}$ linearly independent? (7 points)

   b. Is the set $\{t, t(t+1), t(t-1)\}$ linearly independent? (7 points)

   c. Suppose that $f(t)$ is differentiable and positive for all t. Are $f(t)$ and $tf(t)$ linearly independent? (6 points)

SOLUTION

a. The Wronskian is

$$\begin{vmatrix} 1 & t & 2t^2-1 & 4t^3-3t \\ 0 & 1 & 4t & 12t^2-3 \\ 0 & 0 & 4 & 24t \\ 0 & 0 & 0 & 24 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 4t & 12t^2-3 \\ 0 & 4 & 24t \\ 0 & 0 & 24 \end{vmatrix} = 1 \times 1 \times \begin{vmatrix} 4 & 24t \\ 0 & 24 \end{vmatrix} = 96$$

Since the Wronskian is a nonzero constant, these functions are linearly independent.

b. The Wronskian is

$$\begin{aligned} \begin{vmatrix} t & t(t+1) & t(t-1) \\ 1 & 2t+1 & 2t-1 \\ 0 & 2 & 2 \end{vmatrix} &= t \times \begin{vmatrix} 2t+1 & 2t-1 \\ 2 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} t(t+1) & t(t-1) \\ 2 & 2 \end{vmatrix} \\ &= t(2(2t+1) - 2(2t-1)) - 1(2t(t+1) - 2t(t-1)) \\ &= 4t - 4t \\ &= 0 \end{aligned}$$

Since the Wronskian is zero, these functions are linearly dependent.

c. The Wronskian is

$$\begin{vmatrix} f(t) & tf(t) \\ f'(t) & f(t) + tf'(t) \end{vmatrix} = f^2(t) + tf(t)f'(t) - tf(t)f'(t) = f^2(t) > 0$$

Since the Wronskian is strictly positive, these functions are linearly independent.

5. Answer the following true/false questions

a. The solutions of the differential equation $y'' + 2y' - 1 = 0$ form a vector space.

b. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then A and A^T have the same Reduced Row Echelon Form.

c. If Span$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} = \mathbb{R}^n$ then the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are linearly independent.

d. The basis of a vector space is not unique.

e. If a solution to the system of equations $\mathbf{Ax} = \mathbf{b}$ exists, this implies that the inverse of $\mathbf{A}$ exists.

SOLUTION

a. False. The equation is nonhomogeneous (or $y(t) \equiv 0$ is not a solution).

b. False. RREF$(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, therefore RREF$(A^T) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

c. True. If the vectors were linearly dependent they could not span $\mathbb{R}^n$.

d. True. Any scalar multiple of the vectors in a basis is also a basis over the same vector space.

e. False. $\mathbf{A}$ is not necessarily square.