1. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

a. Find the eigenvalues of **A** (6 points).

b. Find the eigenvectors of **A** (14 points).

SOLUTION

a. Find the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 1-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) - 2 \\ &= \lambda^2 - 4\lambda + 1 \\ &= (\lambda - 2)^2 - 3 \\ &= 0 \end{aligned}$$

Therefore, $\lambda_1 = 2 + \sqrt{3}$ and $\lambda_2 = 2 - \sqrt{3}$.

b. Solve $(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$. Create the augmented matrix and put into RREF.

$$\begin{aligned} &\left(\begin{array}{cc|c} -(1+\sqrt{3}) & -2 & 0 \\ -1 & (1-\sqrt{3}) & 0 \end{array} \right) \\ \to &\left(\begin{array}{cc|c} -(1+\sqrt{3}) & -2 & 0 \\ -(1+\sqrt{3}) & -2 & 0 \end{array} \right) \quad ② \leftarrow ② \times (1+\sqrt{3}) \\ \to &\left(\begin{array}{cc|c} -(1+\sqrt{3}) & -2 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad ② \leftarrow ② - ① \\ \to &\left(\begin{array}{cc|c} 1 & -(1-\sqrt{3}) & 0 \\ 0 & 0 & 0 \end{array} \right) \quad ① \leftarrow ① \times (1-\sqrt{3})/2 \end{aligned}$$

Therefore

$$v_1 = (1-\sqrt{3})\, v_2.$$

So $\mathbf{v}_1$ is

$$\mathbf{v}_1 = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}.$$

Solve $(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}_2 = \mathbf{0}$. Create the augmented matrix and put into RREF.

$$\left(\begin{array}{cc|c} -(1-\sqrt{3}) & -2 & 0 \\ -1 & (1+\sqrt{3}) & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} -(1-\sqrt{3}) & -2 & 0 \\ -(1-\sqrt{3}) & -2 & 0 \end{array}\right) \quad ② \leftarrow ② \times (1-\sqrt{3})$$

$$\rightarrow \left(\begin{array}{cc|c} -(1-\sqrt{3}) & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad ② \leftarrow ② - ①$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & -(1+\sqrt{3}) & 0 \\ 0 & 0 & 0 \end{array}\right) \quad ① \leftarrow ① \times (1+\sqrt{3})/2$$

Therefore

$$v_1 = (1+\sqrt{3})\,v_2.$$

So $\mathbf{v}_2$ is

$$\mathbf{v}_2 = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}.$$

2. (20 points) Solve the following initial value problem

$$y'' + \frac{1}{t}y' - \frac{4}{t^2}y = t^2 \quad y(1) = 1, y'(1) = 2, t > 0$$

using variation of parameters and knowing that the general homogeneous solution is given by

$$y_h(t) = c_1 t^2 + c_2 \frac{1}{t^2}$$

SOLUTION

Using the variation of parameters set

$$y_p(t) = s_1(t) t^2 + s_2(t) \frac{1}{t^2}$$

and

$$y_p'(t) = s_1(t)\ (t^2)' + s_2(t)\ \left(\frac{1}{t^2}\right)' \quad \text{with constraint} \quad s_1'(t)\ (t^2) + s_2'(t)\ \left(\frac{1}{t^2}\right) = 0$$

$$y''{}_p(t) = y_p'(t)$$

This leads to the system

$$\begin{pmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{pmatrix} \begin{pmatrix} s_1' \\ s_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$$

Thus

$$s_1'(t) = \frac{\begin{vmatrix} 0 & t^{-2} \\ t^2 & -2t^{-3} \end{vmatrix}}{\begin{vmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{vmatrix}} \quad s_2'(t) = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & t^2 \end{vmatrix}}{\begin{vmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{vmatrix}}$$

$$s_1'(t) = \frac{-1}{-4t^{-1}} \quad s_2'(t) = \frac{t^4}{-4t^{-1}}$$

$$s_1'(t) = \frac{t}{4} \quad s_2'(t) = -\frac{t^5}{4}$$

$$s_1(t) = \frac{t^2}{8} \quad s_2(t) = -\frac{t^6}{24}$$

Thus

$$y_p(t) = t^2\frac{t^2}{8} - t^{-2}\frac{t^6}{24} = \frac{1}{12}t^4$$

Thus the general solution is given

$$y(t) = c_1t^2 + c_2t^{-2} + \frac{1}{12}t^4$$

The initial value problem gives

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11/12 \\ 5/3 \end{pmatrix}$$

This gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7/8 \\ 1/24 \end{pmatrix}$$

3. Consider the system of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

   a. Find the characteristic equation and eigenvalues for this system (5 points).

   b. Find the eigenvectors for **A** (5 points).

   c. Find the general solution of this system (5 points).

   d. What is the span of the eigenvectors of **A** (5 points)?

SOLUTION

a. The characteristic equation for A is $(\lambda-3)^2(\lambda+3) = 0$ and the eigenvalues are $\lambda_1 = \lambda_2 = 3$ and $\lambda_3 = -3$.

b. We need to solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, where $\mathbf{v} = [v_1, v_2, v_3]$.

For $\lambda = 3$:

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives that $-9v_2 = 0$ or $v_2 = 0$. One can therefore choose the two linearly independent eigenvectors $\mathbf{v}_1 = [1, 0, 0]$ and $\mathbf{v}_2 = [0, 0, 1]$.

For $\lambda = -3$:

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

which gives that $v_1 = v_3 = 0$. One can choose the eigenvector $\mathbf{v}_3 = [0, 1, 0]$.

c. The general solution is

$$\mathbf{x}_g = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

d. The eigenvectors span $\mathbb{R}^3$.

4. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$$

a. Find the eigenvalues of **A** (6 points).

b. Find the eigenvectors of **A** (14 points).

SOLUTION

a. Find the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) + 8 \\ &= \lambda^2 - 4\lambda + 11 \\ &= (\lambda - 2)^2 + 7 \\ &= 0 \end{aligned}$$

Therefore, $\lambda_1 = 2 + i\sqrt{7}$ and $\lambda_2 = 2 - i\sqrt{7}$.

b. Solve $(\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$. Create the augmented matrix and put into RREF.

$$\begin{aligned} &\left(\begin{array}{cc|c} -1-i\sqrt{7} & 2 & 0 \\ -4 & 1-i\sqrt{7} & 0 \end{array}\right) \\ \rightarrow &\left(\begin{array}{cc|c} -4(1+i\sqrt{7}) & 8 & 0 \\ -4(1+i\sqrt{7}) & 8 & 0 \end{array}\right) \quad \begin{array}{l} ① \leftarrow ① \times -4 \\ ② \leftarrow ② \times (1+i\sqrt{7}) \end{array} \\ \rightarrow &\left(\begin{array}{cc|c} -4(1+i\sqrt{7}) & 8 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad ② \leftarrow ② - ① \\ \rightarrow &\left(\begin{array}{cc|c} 1 & -(1-i\sqrt{7})/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad ① \leftarrow ① \times -(1-i\sqrt{7})/32 \end{aligned}$$

Therefore

$$4\,v_1 = (1 - i\sqrt{7})\,v_2.$$

So $\mathbf{v}_1$ is

$$\mathbf{v}_1 = \begin{pmatrix} 1 - i\sqrt{7} \\ 4 \end{pmatrix}.$$

Since $\mathbf{v}_2$ is the complex conjugate of $\mathbf{v}_1$,

$$\mathbf{v}_2 = \begin{pmatrix} 1 + i\sqrt{7} \\ 4 \end{pmatrix}.$$

5. For each of the following equations, determine whether the guess of the particular solution will or will not work using to the method of **undertermined coefficients** (4 points each). Note: you are *not* required to show your work.

   a. $y'' + y = 3\sin(t)$ and $y_p = c_1\cos(t) + c_2\sin(t)$.

   b. $y'' + y = 6\sin(2t)$ and $y_p = c_1\cos(2t) + c_2\sin(2t)$.

   c. $y'' - 4y' + 4y = e^{-2t}$ and $y_p = c_1 e^{-2t}$.

   d. $y'' - 4y' + 4y = te^{-2t}$ and $y_p = c_1 te^{-2t} + c_2 e^{-2t}$.

   e. $y'' - 3y' = 3t^2$ and $y_p = c_1 t^2 + c_2 t + c_3$.

SOLUTION

a. No.

b. Yes.

c. Yes.

d. Yes.

e. No.