
INSTRUCTIONS:

- Computers, calculators, books, notes, and crib sheets are not permitted.
- Write your name, instructor's name, and recitation number on the front of your bluebook.
- Work all **eight problems**. Start each problem on a new page.
- Show your work and clearly identify your final answer.

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1. (25 points) Consider the following first order ODE

$$y' + ty = -e^{t^2} y^3$$

   a. Show that the equation given above can be written as $v' - 2tv = 2e^{t^2}$ using the substitution $v = y^{-2}$. (4 points)

   b. Find the general solution, $v(t)$, of the equation in part a. (10 points)

   c. Find the general solution, $y(t)$, of the original equation. (5 points)

   d. Given the initial condition $y(0) = -1$, solve the IVP. (6 points)

2. (25 points) Consider the following initial-value problems

$$y' = 3\frac{\sin t}{t^2} y^{\frac{2}{3}} \quad y(t_0) = y_0 \tag{1}$$

$$y'' + (\sin t)y' + (\cos t)y = t^2 \quad y(t_0) = y_0 \ \ y'(t_0) = y_0' \tag{2}$$

   where t_0, y_0 and y_0' are arbitrary constants.

   a. For equation (1) indicate the regions in the (t_0, y_0) plane for which there is a unique solution. (10 points)

   b. For equation (1) with $t_0 = y_0 = 1$ describe the interval over which the solution extends (do *not* solve the equation)? (2 points)

   c. For equation (2) indicate the regions in the (t_0, y_0) plane for which there is a unique solution. (10 points)

   d. For equation (2) with $t_0 = y_0 = 1$ describe the interval over which the solution extends (do *not* solve the equation)? (2 points)

3. (25 points) Consider the following IVP

$$y'' - 4y' + 4y = \frac{e^{2t}}{t}, \ y(1) = 0, \ y'(1) = 0, \ t > 0$$

   a. Find the general solution to the homogeneous equation. (7 points)

   b. Find the particular solution. (7 points)

   c. Find the general solution. (4 points)

   d. Solve the IVP. (7 points)

4. (25 points) Consider the second order ODE

$$y'' - yy' + y = 0$$

a. Convert this equation to an equivalent system of first order ODE's. (10 points)

b. In the (y, y') phase plane, sketch the nullclines, equilbrium point(s) for $y' \le 1$. (10 points)

c. In the open regions of your phase-plane sketch for $y' \le 1$ indicate the flow direction of the solution trajectories. (5 points)

5. (25 points) Consider the system of equations $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

a. Find the characteristic equation and eigenvalues for this system (5 points).

b. Find the eigenvectors for A (5 points).

c. Find the general solution of this system (5 points).

d. What do the eigenvectors of A span (5 points)?

6. Answer the following True/False questions (5 points each):

a. The set $V = Span\{1, x, \tan(x^2)\}$ is a vector space.

b. $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

c. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $A = \begin{bmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{bmatrix}$. Then $\lambda_1\lambda_2\lambda_3 = 0$ and

$\lambda_1 + \lambda_2 + \lambda_3 = -(a + e)$.

d. Let $\dot{\mathbf{x}} = A\mathbf{x}$ be a 3x3 system of linear differential equations with constant real coefficients. If $\mathbf{x} = \mathbf{0}$ is a saddle point then the eigenvalues of A must be real.

e. The system $\begin{bmatrix} 1 & -2 \\ -3 & k \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ has a nonzero solution if and only if $k = 6$.

7. (25 points) Consider the following system of ODE's

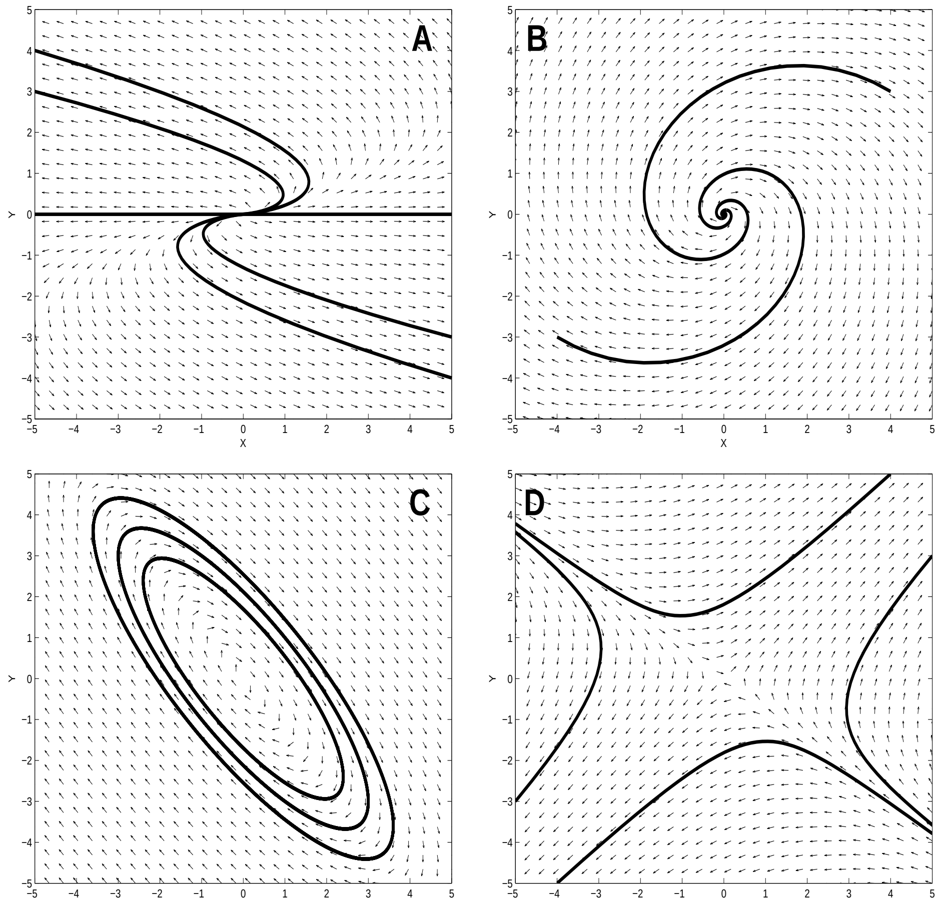
$$\begin{aligned} \dot{x} &= x - x^2 - xy \\ \dot{y} &= \tfrac{3}{4}y - y^2 - \tfrac{1}{2}xy \end{aligned}$$

for $x, y \ge 0$.

a. Find all the equilibrium points. (5 points)

b. Find the linearized system that approximates the solution behavior in the neighborhood of each equilibrium points. (10 points)

c. Classify the equilibrium points. (10 points)



8. (25 points) For each of the following systems determine if the system matches one of the phase portraits given above and, if so, state which one.

$$\text{(i) } \mathbf{x}' = \begin{pmatrix} 2 & 2 \\ -3 & -2 \end{pmatrix} \mathbf{x}$$

$$\text{(ii) } \mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{x}$$

$$\text{(iii) } \mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \mathbf{x}$$

$$\text{(iv) } \mathbf{x}' = \begin{pmatrix} 1 & 4 \\ -4 & -2 \end{pmatrix} \mathbf{x}$$

$$\text{(v) } \mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \mathbf{x}$$