
INSTRUCTIONS:

- Computers, calculators, books, notes, and crib sheets are not permitted.
- Write your name, instructor's name, and recitation number on the front of your bluebook.
- Work all **eight problems**. Start each problem on a new page.
- Show your work and clearly identify your final answer.

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1. (25 points) Consider the following first order ODE

$$y' + ty = -e^{t^2} y^3$$

a. Show that the equation given above can be written as $v' - 2tv = 2e^{t^2}$ using the substitution $v = y^{-2}$. (4 points)

b. Find the general solution, $v(t)$, of the equation in part a. (10 points)

c. Find the general solution, $y(t)$, of the original equation. (5 points)

d. Given the initial condition $y(0) = -1$, solve the IVP. (6 points)

SOLUTION

a. We need to find an expression for $y(t)$ and $y'(t)$ in terms of $v(t)$. The derivative of $v(t)$ gives

$$v' = -2y'y^{-3},$$

which can also be written as

$$y' = -\frac{1}{2}v'y^3.$$

The original substitution can also be written as

$$y = vy^3.$$

Now, substituting for $y(t)$ in the original equation gives

$$-\frac{1}{2}v'y^3 + tvy^3 = -e^{t^2}y^3,$$

or,

$$v' - 2tv = 2e^{t^2}.$$

b. We consider two solution methods: the integrating factor method and variation of parameters. First, the integrating factor method. We want a function $\mu(t)$ such that

$$\mu(v' - 2tv) = (\mu v)'.$$

Using the product rule on the RHS and rearranging gives

$$v(\mu' + 2t\mu) = 0.$$

Since $v(t) \neq 0$ (this would not satisfy the original equation), this implies

$$\mu' + 2t\mu = 0.$$

Therefore the integrating factor is

$$\mu(t) = e^{-t^2}.$$

So the differential equation can be written as

$$(\mu v)' = 2\mu e^{t^2}.$$

Directly integrating gives,

$$v = (2t + A)e^{t^2},$$

where A is the constant of integration.

Using the variation of pararmeters method, we first solve the homogeneous equation,

$$v_h' - 2tv_h = 0.$$

This is simply

$$v_h(t) = Ae^{t^2},$$

where A is the constant of integration. Now we propose that the particular solution has the form

$$v_p(t) = u(t)e^{t^2},$$

where $u(t)$ is an unknown function. Substituting $v_p(t)$ into the differential equation gives

$$\begin{aligned} (ue^{t^2})' - 2t(ue^{t^2}) &= 2e^{t^2} \\ u'e^{t^2} + 2tue^{t^2} - 2tue^{t^2} &= 2e^{t^2} \\ u'e^{t^2} &= 2e^{t^2} \\ u' &= 2. \end{aligned}$$

Therefore,

$$u(t) = 2t.$$

The general solution is the sum of the particular and homogeneous solutions

$$v(t) = (2t + A)e^{t^2}.$$

c. The general solution to the original equation is obtained by substituting the solution $v(t)$ into the original substitution, i.e.

$$y(t) = \pm\frac{e^{-t^2/2}}{\sqrt{2t+A}}.$$

It is important to recognise that there are two solutions.

d. The initial condition implies that the solution to the IVP must be of the form

$$y(t) = -\frac{e^{-t^2/2}}{\sqrt{2t+A}}.$$

Substituting in the initial values gives

$$-1 = -\frac{1}{\sqrt{A}}.$$

Therefore, $A = 1$, so that the solution to the IVP is

$$y(t) = -\frac{e^{-t^2/2}}{\sqrt{2t+1}}$$

2. (25 points) Consider the following initial-value problems

$$y' = 3\frac{\sin t}{t^2}y^{\frac{2}{3}} \quad y(t_0) = y_0 \tag{1}$$

$$y'' + (\sin t)y' + (\cos t)y = t^2 \quad y(t_0) = y_0 \;\; y'(t_0) = y_0' \tag{2}$$

where t_0, y_0 and y_0' are arbitrary constants.

a. For equation (1) indicate the regions in the (t_0, y_0) plane for which there is a unique solution. (10 points)

b. For equation (1) with $t_0 = y_0 = 1$ describe the interval over which the solution extends (do *not* solve the equation)? (2 points)

c. For equation (2) indicate the regions in the (t_0, y_0) plane for which there is a unique solution. (10 points)

d. For equation (2) with $t_0 = y_0 = 1$ describe the interval over which the solution extends (do *not* solve the equation)? (2 points)

SOLUTION

a. The region of acceptable points must not include the lines $t = 0$ ($f(t, y) = \infty$) and $y \leq 0$ ($f_y(t, y) = \infty$).

b. The ODE is nonlinear so the existence and uniqueness theorem guarantees a range $(t_0 - \varepsilon, t_0 + \varepsilon)$ with *varepsilon* $\ll 1$.

c. The region of acceptable points is *calR*^{3} for the space t, y, y'.

d. The ODE is linear so the existence and uniqueness theorem guarantees a range $\forall t$.

3. (25 points) Consider the following IVP

$$y'' - 4y' + 4y = \frac{e^{2t}}{t},\ y(1) = 0,\ y'(1) = 0,\ t > 0$$

a. Find the general solution to the homogeneous equation. (7 points)

b. Find the particular solution. (7 points)

c. Find the general solution. (4 points)

d. Solve the IVP. (7 points)

SOLUTION

a. The homogeneous equation is

$$y_h'' - 4y_h' + 4y_h = 0$$

Assuming a solution of the form $y(t) = e^{\lambda t}$ gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0.$$

So the characteristic equation has the repeated root $\lambda = 2$. Therefore, the general solution has the form

$$y_h(t) = a_1 y_1(t) + a_2 y_2(t),$$

where $a_1, a_2 \in \mathbb{R}$, $y_1(t) = e^{2t}$, and $y_2(t) = te^{2t}$.

b. Use the variation of parameters method to determine the particular solution. Suppose that the particular solution can be written in the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

where $v_1(t)$ and $v_2(t)$ are unknown functions. Differentiating y_p gives

$$y_p' = v_1'y_1 + v_2'y_2 + v_1y_1' + v_2y_2'.$$

Differentiating y_p twice gives

$$y_p'' = v_1''y_1 + v_2''y_2 + 2v_1'y_1' + 2v_2'y_2' + v_1y_1'' + v_2y_2''.$$

Hence,

$$\begin{aligned} y_p'' - 4y_p' + 4y_p &= v_1''y_1 + v_2''y_2 + 2v_1'y_1' + 2v_2'y_2' + v_1y_1'' + v_2y_2'' \\ &\quad -4(v_1'y_1 + v_2'y_2 + v_1y_1' + v_2y_2') + 4(v_1y_1 + v_2y_2). \end{aligned}$$

The v_1 terms and the v_2 terms are a part of the homogeneous problem and sum to zero leaving

$$y_p'' - 4y_p' + 4y_p = v_1''y_1 + v_2''y_2 + 2v_1'y_1' + 2v_2'y_2' - 4(v_1'y_1 + v_2'y_2),$$

which can be rewritten as

$$y_p'' - 4y_p' + 4y_p = (v_1'y_1 + v_2'y_2)' + v_1'y_1' + v_2'y_2' - 4(v_1'y_1 + v_2'y_2).$$

Now we introduce the auxilliary condition

$$v_1'y_1 + v_2'y_2 = 0,$$

which implies that

$$v_1'y_1' + v_2'y_2' = t^{-1}e^{2t}.$$

Solving this system of equations for v_1' and v_2' using Cramer's rule gives

$$\begin{aligned} v_1' &= \frac{-y_2\, t^{-1}e^{2t}}{W(y_1, y_2)}, \\ v_2' &= \frac{y_1\, t^{-1}e^{2t}}{W(y_1, y_2)}, \end{aligned}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{4t}.$$

This gives

$$\begin{aligned} v_1' &= -1, \\ v_2' &= t^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} v_1 &= -t \\ v_2 &= \log(t), \text{ since } t > 0. \end{aligned}$$

Since $v_1y_1 = -te^{2t}$ is one of the terms in the solution to the homogeneous problem, it does not need to be included in the particular solution. So the particular solution is

$$y_p(t) = t\log(t)\, e^{2t}.$$

c. The general solution is the sum of the general solution to the homogeneous problem and the particular solution,

$$y(t) = (a_1 + a_2 t + t\log(t))e^{2t}.$$

d. First, determine the derivative of the general solution

$$y'(t) = e^{2t}\big(2a_1 + a_2 + 2a_2 t + (t+1)\log(t) + 1\big).$$

Now substituting in the initial values for $y(t)$ and $y'(t)$ gives

$$\begin{aligned} a_1 + a_2 &= 0, \\ 2a_1 + 3a_2 + 1 &= 0. \end{aligned}$$

Solving for a_1 and a_2

$$\begin{aligned} a_1 &= 1, \\ a_2 &= -1. \end{aligned}$$

Therefore, the solution to the IVP is

$$y(t) = (1 - t + t\log(t))e^{2t}$$

4. (25 points) Consider the second order ODE

$$y'' - yy' + y = 0$$

a. Convert this equation to an equivalent system of first order ODE's. (10 points)

b. In the (y, y') phase plane, sketch the nullclines, equilbrium point(s) for $y' \leq 1$. (10 points)

c. In the open regions of your phase-plane sketch for $y' \leq 1$ indicate the flow direction of the solution trajectories. (5 points)

5. (25 points) Consider the system of equations $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

a. Find the characteristic equation and eigenvalues for this system (5 points).

b. Find the eigenvectors for A (5 points).

c. Find the general solution of this system (5 points).

d. What do the eigenvectors of A span (5 points)?

SOLUTION

a. The characteristic equation for A is $(\lambda+2)^2(\lambda-2)=0$ and the eigenvalues are $\lambda_1=\lambda_2=-2$ and $\lambda_3=2$.

b. We need to solve $(A-\lambda I)\mathbf{v}=\mathbf{0}$, where $\mathbf{v}=[v_1,v_2,v_3]$.
For $\lambda_3=2$:

$$A-2I=\begin{bmatrix}-4 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & -4\end{bmatrix}.$$

Therefore, one can choose the eigenvector $\mathbf{v}_3=[0,1,0]$.
For $\lambda=-2$:

$$A+2I=\begin{bmatrix}0 & 0 & 1\\ 0 & 4 & 0\\ 0 & 0 & 0\end{bmatrix},$$

which gives that $v_2=v_3=0$. Therefore, one can choose the eigenvector $\mathbf{v}_2=[1,0,0]$.
We need, in addition, a generalized eigenvector for this eigenvalue, so we solve $(A+2I)\mathbf{u}=\mathbf{v}_2$, i.e.,

$$\left[\begin{array}{ccc|c}0 & 0 & 1 & 1\\ 0 & 4 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$

which gives that $u_3=1$ and $u_2=0$. One can therefore choose $\mathbf{u}=[0,0,1]$.

c. The general solution is

$$\mathbf{x}_g=c_1e^{2t}\begin{bmatrix}0\\1\\0\end{bmatrix}+e^{-2t}\left(c_2\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+e^{-2t}c_3\left(t\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}\right).$$

d. Since $\mathbf{u}$ is *not* an eigenvector, the eigenvectors of A span a two dimensional subspace of $\mathbb{R}^3$.

6. Answer the following True/False questions (5 points each):

a. The set $V=Span\{1,x,\tan(x^2)\}$ is a vector space.

b. $\begin{bmatrix}2\\0\end{bmatrix}$ is an eigenvector of $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$.

c. Let $\lambda_1,\lambda_2,\lambda_3$ be the eigenvalues of $A=\begin{bmatrix}a & b & c\\ 0 & 0 & d\\ 0 & 0 & e\end{bmatrix}$. Then $\lambda_1\lambda_2\lambda_3=0$ and

$\lambda_1+\lambda_2+\lambda_3=-(a+e)$.

d. Let $\dot{\mathbf{x}} = A\mathbf{x}$ be a 3x3 system of linear differential equations with constant real coefficients. If $\mathbf{x} = \mathbf{0}$ is a saddle point then the eigenvalues of A must be real.

e. The system $\begin{bmatrix} 1 & -2 \\ -3 & k \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ has a nonzero solution if and only if $k = 6$.

SOLUTION

a. Yes. A span forms a vector space.

b. Yes.

c. No. $\lambda_1 + \lambda_2 + \lambda_3 = a + e$.

d. No. If it's a saddle point then $|A| < 0$.

e. No. It has a (zero and nonzero) solution iff $k \neq 6$.

7. (25 points) Consider the following system of ODE's

$$\begin{aligned} \dot{x} &= x - x^2 - xy \\ \dot{y} &= \tfrac{3}{4}y - y^2 - \tfrac{1}{2}xy \end{aligned}$$

for $x, y \geq 0$.

a. Find all the equilibrium points. (5 points)

b. Find the linearized system that approximates the solution behavior in the neighborhood of each equilibrium points. (10 points)

c. Classify the equilibrium points. (10 points)

SOLUTION

a. Equilibrium points: $(0,0), (0,3/4), (1,0), (1/2,1/2)$

b. The linearised systems have the form $\dot{\mathbf{u}} = \mathbf{J}\mathbf{u}$ where

$$\mathbf{J} = \begin{pmatrix} 1 - 2x - y & -x \\ -\frac{1}{2}y & \frac{3}{4} - 2y - \frac{1}{2}x \end{pmatrix}$$

The specfic Jacobians are

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}_{(0,0)} \quad \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{pmatrix}_{(0,3/4)} \quad \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{pmatrix}_{(1,0)} \quad \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}_{(1/2,1/2)}$$

c. The corresponding eigenvalues $(\lambda_1, \lambda_2)$are $(1, 3/4), (1/4, -3/4), (-1, 1/4), (-(2-\sqrt(2))/4, , -(2+\sqrt(2))/4)$ so the equilibrium points are respectively (a source, saddle, saddle, sink).

8. (25 points) For each of the following systems determine if the system matches one of the phase portraits given above and, if so, state which one.

$$\text{(i) } \mathbf{x}' = \begin{pmatrix} 2 & 2 \\ -3 & -2 \end{pmatrix} \mathbf{x}$$

$$\text{(ii) } \mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{x}$$

$$\text{(iii) } \mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \mathbf{x}$$

$$\text{(iv) } \mathbf{x}' = \begin{pmatrix} 1 & 4 \\ -4 & -2 \end{pmatrix} \mathbf{x}$$

$$\text{(v) } \mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \mathbf{x}$$

SOLUTION

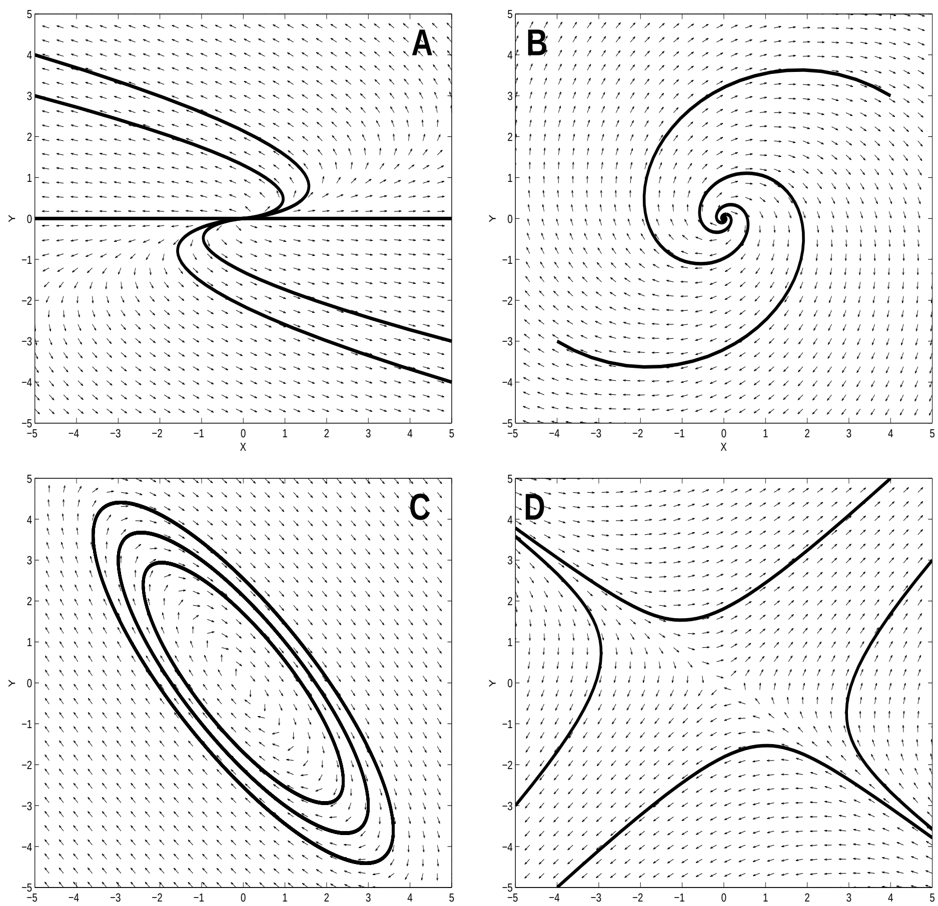
i This system has a pair of purely complex conjugate eigenvalues corresponding to the **center** (C).

ii This system has a pair of distinct, real eigenvalues that have different signs. So this system corresponds to the **saddle node** (D).

iii This system has a pair of repeated eigenvalues and one eigenvector giving the **repelling degenerate node** (A).

iv This system has a pair of complex conjugate eigenvalues with a positive real part. This system corresponds to the **repelling spiral** (B).

v This system has a pair of repeated eigenvalues corresponding to an **attracting degenerate node**, so it does not match any of the phase portraits.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