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**INSTRUCTIONS:**

- Computers, calculators, books, notes, and crib sheets are not permitted.
- Write your name, instructor's name, and recitation number on the front of your bluebook.
- Work all **five problems**. Start each problem on a new page.
- Show your work and clearly identify your final answer.

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1. (20 points)

   Classify each of the following differential equations by (i) order, (ii) linear or nonlinear, (iii) separable or not, and *if* it is linear (iv) homogeneous or nonhomogeneous.

   $(a)\;\; y' + t^2y'' = y - t^3 \qquad (b)\;\; y' = e^y t - 3 \qquad (c)\;\; y' - ty = 3y \qquad (d)\;\; y' + y'' = t\sin(y)$

1. **Solution** (a) (i) order =2 (ii) linear (iii) NOT seperable (iv) nonhomogeneous

   (b) (i) order =1 (ii) nonlinear (iii) NOT seperable (iv) N/A

   (c) (i) order=1 (ii) linear (iii) seperable (iv) homogeneous

   (d) (i) order =2 (ii) nonlinear (iii) NOT seperable (iv) N/A

2. (20 points)

   Mark each of the following statements as True or False. (No work need be shown).

   a) $L(x, y) = 3y + 4 + x$ is a linear operator.

   b) Picard's theorem implies that $y' = \sqrt{y} + t^{1/3}, \quad y(0) = 3$ has a unique solution.

   c) $y(t) = \alpha t$ is a solution of $ty' + 4y = 5t$ for $\alpha = 1$.

   d) $y(t) = \sin(t)$ is an equilibrium solution of $y' = \cos(t)$.

   e) If $y_1,\ y_2$ are solutions of $y'' + e^t y = e^{-t^2}$ then $y = y_1 - y_2$ is a solution of $y'' + e^t y = 0$.

2. **Solution** (a) FALSE: $x^0$ or $y^0$ not allowed. $L(x, y) + 4 = 3y + x + 4$ would make $L$ a linear operator

   (b) TRUE: Let $\mathcal{R} = \{(t, y)|t \in (-\infty, \infty), y \in (0, \infty)\}$. Then $f(t, y) = \sqrt{y} + t^{1/3}$ is continuous on $\mathcal{R}$, $f_y$ is continuous on $\mathcal{R}$ and $(t_0, y_0) = (0, 3) \in \mathcal{R}$.

   (c) TRUE: substitution

   (d) FALSE: $y(t) = \sin(t)$ is a solution, but is not a constant function.

   (e) TRUE: $y'' + e^t y = e^{-t^2}$ is a 2nd order linear nonhomogenous ODE. Thus, the *nonhomogeneous principle* applies. Thus $y_{p1} = y_h + y_{p2}$, where $y_h = y$ in this case. Or, one could subsititute $y = y_1 - y_2$ into $y'' + e^t y = 0$ and show that it is true.

3. (25 points)

   a) Find the homogeneous solution of $y' + ty = t$.

   b) Find the particular solution of $y' + ty = t$.

   c) Given $y(0) = 2$, what is the solution of $y' + ty = t$?

3. **Solution** a)

$$y' + ty = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -ty$$
$$\Rightarrow \ln(|y|) = -\frac{1}{2}t^2 \quad \Rightarrow \quad y_h(t) = Ce^{-\frac{1}{2}t^2}$$

b)

$$y' + ty = t$$
$$\text{use } p(t) = t \text{ and } f(t) = t$$
$$\Rightarrow \mu(t) = e^{\int_0^t \hat{t}d\hat{t}} = e^{\frac{1}{2}t^2}$$
$$\begin{aligned} y_p(t) &= \frac{1}{\mu(t)}\int_0^t \mu(\hat{t})f(\hat{t})d\hat{t} \\ &= e^{-\frac{1}{2}t^2}\int_0^t e^{\frac{1}{2}\hat{t}^2}\hat{t}d\hat{t} \\ \text{let } u = \frac{1}{2}t^2 \Rightarrow \quad &= e^{-\frac{1}{2}t^2}\int_0^{\frac{1}{2}t^2} e^u du \\ &= e^{-\frac{1}{2}t^2}(e^{\frac{1}{2}t^2} - 1) \end{aligned}$$
$$\Rightarrow \boxed{y_p(t) = 1 - e^{-\frac{1}{2}t^2}}$$

c)

$$y_g(t) = Ce^{-\frac{1}{2}t^2} + 1$$
$$y(0) = 2 \Rightarrow C + 1 = 2 \Rightarrow C = 1$$
$$\Rightarrow \boxed{y(t) = e^{-\frac{1}{2}t^2} + 1}$$

4. (15 points)

a) Given $y'(t) = f(t, y)$, write the expression for $t_{n+1}$ and $y_{n+1}$ in terms of $y_n, t_n, h$ for Euler's numerical method.

b) Use Euler's Method to find the numerical solution for 2 steps (ie. $y_1, y_2$) to the problem

$$\begin{aligned} y' &= y^2 - 6y + 5 \\ y(0) &= 3 \\ \text{using } h &= 0.25 \text{ as your step size.} \end{aligned}$$

4. **Solution** a)

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

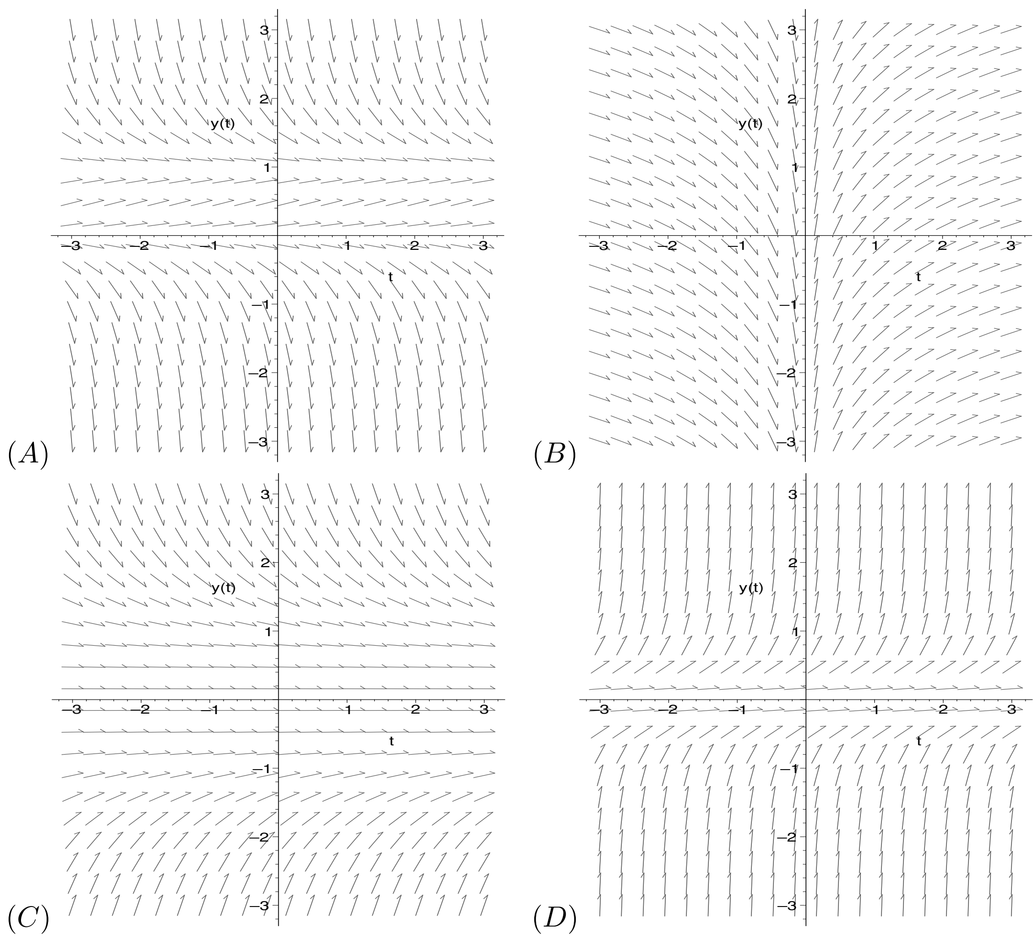
b)

$$y_1 = 3 + \frac{1}{4}(9 - 18 + 5)$$
$$\boxed{y_1 = 2}$$
$$y_2 = 2 + \frac{1}{4}(4 - 12 + 5)$$
$$\boxed{y_2 = \frac{5}{4}}$$

5. (20 points)

   a) Match each of the direction fields (A)-(D) with one of the following differential equations :

$$(1)\ \ y' = \frac{1}{y} \qquad (2)\ \ y' = y\cdot(1-y) \qquad (3)\ \ y' = 3y^2 \qquad (4)\ \ y' + y = \sin(y) \qquad (5)\ \ y' = t^3$$




   b) For equation (4), suppose $y(0) = 2$. What will be $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

   c) For equation (2), what are the equilibria?

   d) Classify the equilibria of equation (2) as stable, unstable or neither.

5. **Solution** a)

$$(A) \Rightarrow (2)$$
$$(B) \Rightarrow (1)$$
$$(C) \Rightarrow (4)$$
$$(D) \Rightarrow (3)$$

b)

$$\begin{aligned}\lim_{t\to\infty} y(t) &= y \ \text{ s.t. } y = \sin(y), 0 \le y \le 2 \ \text{ from dir. field} \\ &\Rightarrow \boxed{\lim_{t\to\infty} y(t) = 0}\end{aligned}$$

and

$$\boxed{\lim_{t\to-\infty} y(t) = \infty}$$

c)

$$y' = y(1-y)$$
$$\Rightarrow \boxed{y = 0 \ \text{ and } \ y = 1 \ \text{ are equilibria.}}$$

d)

$$\boxed{y = 0 \text{ is unstable }, y = 1 \text{ is stable.}}$$