
INSTRUCTIONS:

- Computers, calculators, books, notes, and crib sheets are not permitted.
 - Write your name, instructor's name, and recitation number on the front of your bluebook.
 - Work all **five problems**. Start each problem on a new page.
 - Show your work and clearly identify your final answer.
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1. (20 points) Initially, you have a tank containing 60 litres of pure water. Water containing salt is pumped into the tank at 1 L/s with a concentration of 2 kg/L. Water from the tank is pumped out at 2 L/s. The water in the tank is continuously mixed so that the concentration of salt in the tank is uniform.
 - a. Write an initial value problem describing the amount of salt in the tank. (5 points)
 - b. Solve the initial value problem from part (a). (12 points)
 - c. What is the concentration of salt in the tank after 30 seconds? (3 points)

2. (20 points) Given the system

$$\begin{aligned}\frac{du}{dt} &= (u - v)^2 \\ \frac{dv}{dt} &= u^2 - v\end{aligned}$$

answer the following questions *for the region* $u \geq 0$ *and* $v \geq 0$.

- a. Calculate the nullclines of this system. (4 points)
- b. Find the equilibrium point(s). (4 points)
- c. Draw the phase portrait of the system showing (i) the nullclines, (ii) the equilibrium point(s), (iii) the solution directions in the open regions, and (iv) the solution directions on the nullclines. (12 points)

3. (20 points) The following system of equations rotates a vector (x_1, x_2, x_3) counterclockwise through an angle θ in the xy -plane and a scaling in the z -direction to produce the vector (u_1, u_2, u_3) .

$$\begin{aligned}x_1 \cos \theta - x_2 \sin \theta &= u_1 \\x_1 \sin \theta + x_2 \cos \theta &= u_2 \\2x_3 &= u_3\end{aligned}$$

- a. Write the system in the form $\mathbf{Ax} = \mathbf{u}$. (6 points)
b. Solve for \mathbf{x} using Cramer's rule. (14 points)
4. (20 points) Consider the system

$$\begin{aligned}x_2 + x_3 &= 1 \\px_1 + qx_3 + x_4 &= 1 \\x_1 + x_2 &= 1\end{aligned}$$

where p and q are constants.

- a. Write the system in matrix form. (6 points)
b. Write the matrix equation in row reduced echelon form. (14 points)
5. (20 points) Answer the following True/False questions.
- a. If \mathbf{A} , \mathbf{B} , and \mathbf{C} are matrices such that $\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}\mathbf{C}^T$, then $\mathbf{M}^T = \mathbf{C}\mathbf{B}^T(\mathbf{A}^T)^{-1}$.
b. If a solution to the system of equations $\mathbf{Ax} = \mathbf{b}$ exists, this implies that the inverse of \mathbf{A} exists.
c. The RREF of a matrix \mathbf{A} is unique.
d. The set of all 2×2 matrices with the diagonal elements equal to 1 is a vector space.
e. If $\det(\mathbf{A}) \neq 0$ then \mathbf{A}^{-1} exists.