
INSTRUCTIONS:

- Computers, calculators, books, notes, and crib sheets are not permitted.
- Write your name, instructor's name, and recitation number on the front of your bluebook.
- Work all **five problems**. Start each problem on a new page.
- Show your work and clearly identify your final answer.

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1. (20 points) Initially, you have a tank containing 60 litres of pure water. Water containing salt is pumped into the tank at 1 L/s with a concentration of 2 kg/L. Water from the tank is pumped out at 2 L/s. The water in the tank is continuously mixed so that the concentration of salt in the tank is uniform.

   a. Write an initial value problem describing the amount of salt in the tank. (5 points)
   b. Solve the initial value problem from part (a). (12 points)
   c. What is the concentration of salt in the tank after 30 seconds? (3 points)

   SOLUTION

   a. The general conservation law is given by

   $$x' = c_1u_1 - c_2u_2$$

   (2 points) where x is the amount of salt in the tank, c_1 is the concentration of salt being pumped into the tank with a flowrate u_1, and c_2 is the concentration of the salt being pumped out of the tank with a flowrate of u_2. c_1, u_1, and u_2 are given in the description. c_2 is unknown but can be described as

   $$c_2 = \frac{x}{V(t)}$$

   where $V(t)$ is the volume of water in the tank at time t. Since water is being pumped out of the tank faster than water is being pumped in, the volume of water in the tank is changing over time. The volume at time t is

   $$V(t) = V_0 - (u_1 - u_2)t$$

   where V_0 is the initial volume of water in the tank. So the differential equation describing this situation is

   $$x' = 2 - \frac{2x}{60 - t}, t < 60$$

   (2 points) The initial condition for the IVP is $x(0) = 0$ (1 point).

b. Variation of parameters:

Begin by solving the homogeneous problem,

$$x_h' = -\frac{2x_h}{60-t}.$$

Using separation of variables gives

$$\ln|x_h| = 2\ln|60-t| + C_1$$

where C_1 is the constant of integration. Since $t < 60$ and $x(t)$ must be positive we can remove the absolute symbols

$$\ln x_h = 2\ln(60-t) + C_1.$$

Exponentiating gives

$$x_h = C_2(60-t)^2$$

where $C_2 = e^{C_1}$ (5 points). To get the particular solution, we try a solution of the form,

$$x_p = v(t)(60-t)^2.$$

Substituting x_p into the differential equation in part (a) gives

$$v' = \frac{2}{(60-t)^2}.$$

Integrating to get $v(t)$,

$$v(t) = \frac{2}{60-t} + C$$

where C is the constant of integration. However, since the constant part of $v(t)$ only duplicates the homogeneous solution, we ignore C, which gives

$$x_p = 2(60-t).$$

(5 points) So the general solution is

$$x(t) = 2(60-t) + C_2(60-t)^2, t < 60.$$

To solve the IVP, C_2 needs to be determined using the initial condition: $x(0) = 0$

$$0 = 2 \times 60 + C_2 \times 60^2 \Rightarrow C_2 = \frac{-1}{30}.$$

So the solution to the IVP is

$$x(t) = 2(60-t) - \frac{1}{30}(60-t)^2, t < 60$$

(2 points).

Integrating factor method:

First, write the equation in the form

$$x' + \frac{2x}{60-t} = 2.$$

Now we need a function, $\mu(t)$, such that

$$\mu\big(x' + \frac{2x}{60-t}\big) = (\mu x)'.$$

Expanding the RHS of this expression using the chain rule and simplifying gives

$$\mu' = \frac{2\mu}{60-t}.$$

The solution to this DE is

$$\mu = C_1\,(60-t)^{-2},$$

(5 points) where C is the constant on integration. However, we can ignore the constant since it contributes nothing to the final solution. Multiplying the first equation by μ, we can write

$$\big((60-t)^{-2}\,x\big)' = 2\,(60-t)^{-2}.$$

Integrating directly gives,

$$(60-t)^{-2}\,x = 2(60-t)^{-1} + C_2,$$

where C_2 is the constant of integration. Finally, multiply by $(60-t)^2$ gives the general solution,

$$x = 2(60-t) + C_2\,(60-t)^2.$$

(5 points) To solve the IVP, we use the initial condition $x(0) = 0$ and the general solution to get,

$$0 = 2 \times 60 + C_2 \times 60^2 \Rightarrow C_2 = \frac{-1}{30}.$$

So the solution to the IVP is

$$x(t) = 2(60-t) - \frac{1}{30}(60-t)^2, t < 60.$$

(2 points)

c. Evalute the solution from part (b) at $t = 30$ s.

$$\begin{aligned} x(30) &= 2(60-30) - \frac{1}{30}(60-30)^2 \\ &= 60-30 \\ &= 30. \end{aligned}$$

So the amount of salt in the tank after 30 seconds is 30 kg. The volume of water in the tank at this time is 30 litres, so the concentration of salt is 1 kg/L (3 points).

2. (20 points) Given the system

$$\frac{du}{dt} = (u-v)^2$$
$$\frac{dv}{dt} = u^2 - v$$

answer the following questions *for the region* $u \geq 0$ *and* $v \geq 0$.

a. Calculate the nullclines of this system. (4 points)

b. Find the equilibrium point(s). (4 points)

c. Draw the phase portrait of the system showing (i) the nullclines, (ii) the equilibrium point(s), (iii) the solution directions in the open regions, and (iv) the solution directions on the nullclines. (12 points)

a) Horizontal nullcline is v=u^2. Vertical nullcline is v = u. b) Equilibria at (0,0) and (1,1)

3. (20 points) The following system of equations rotates a vector (x_1, x_2, x_3) counterclockwise through an angle θ in the xy-plane and a scaling in the z-direction to produce the vector (u_1, u_2, u_3).

$$\begin{aligned} x_1 \cos\theta - x_2 \sin\theta &= u_1 \\ x_1 \sin\theta + x_2 \cos\theta &= u_2 \\ 2x_3 &= u_3 \end{aligned}$$

a. Write the system in the form $\mathbf{Ax} = \mathbf{u}$. (6 points)

b. Solve for $\mathbf{x}$ using Cramer's rule. (14 points)

SOLUTION

a. The system can be written as $\mathbf{Ax} = \mathbf{u}$ where

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(6 points).

b. According to Cramer's rule,

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}}$$

so we need $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$. These matrices are

$$\mathbf{A}_1 = \begin{bmatrix} u_1 & -\sin\theta & 0 \\ u_2 & \cos\theta & 0 \\ u_3 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \cos\theta & u_1 & 0 \\ \sin\theta & u_2 & 0 \\ 0 & u_3 & 2 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} \cos\theta & -\sin\theta & u_1 \\ \sin\theta & \cos\theta & u_2 \\ 0 & 0 & u_3 \end{bmatrix}.$$

Their determinants are

$$\begin{aligned} \det \mathbf{A}_1 &= \begin{vmatrix} u_1 & -\sin\theta & 0 \\ u_2 & \cos\theta & 0 \\ u_3 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} u_1 & -\sin\theta \\ u_2 & \cos\theta \end{vmatrix} = 2(u_1\cos\theta + u_2\sin\theta), \\ \det \mathbf{A}_2 &= \begin{vmatrix} \cos\theta & u_1 & 0 \\ \sin\theta & u_2 & 0 \\ 0 & u_3 & 2 \end{vmatrix} = 2 \begin{vmatrix} \cos\theta & u_1 \\ \sin\theta & u_2 \end{vmatrix} = 2(u_2\cos\theta - u_1\sin\theta), \\ \det \mathbf{A}_3 &= \begin{vmatrix} \cos\theta & -\sin\theta & u_1 \\ \sin\theta & \cos\theta & u_2 \\ 0 & 0 & u_3 \end{vmatrix} = u_3 \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = u_3, \end{aligned}$$

(3 points each) and the determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 2.$$

(3 points) therefore the solution vector is

$$\mathbf{x} = \begin{bmatrix} u_1\cos\theta + u_2\sin\theta \\ -u_1\sin\theta + u_1\cos\theta \\ u_3/2 \end{bmatrix}$$

(3 points).

4. (20 points) Consider the system

$$\begin{aligned} x_2 + x_3 &= 1 \\ px_1 + qx_3 + x_4 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

where p and q are constants.

a. Write the system in matrix form. (6 points)

b. Write the matrix equation in row reduced echelon form. (14 points)

SOLUTION

a. The system can be written as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ p & 0 & q & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(6 points).

b. Form the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 1 \\ p & 0 & q & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Swap rows 1 and 3.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ p & 0 & q & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Swap rows 2 and 3.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ p & 0 & q & 1 & 1 \end{array}\right]$$

Eliminate element $(3, 1)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -p & q & 1 & 1-p \end{array}\right]$$

Eliminate element $(3, 2)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & q+p & 1 & 1 \end{array}\right]$$

(8 points). If $p + q \neq 0$, then let $r = (p + q)^{-1}$ and RREF($\mathbf{A}$) is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & r & r \\ 0 & 1 & 0 & -r & 1-r \\ 0 & 0 & 1 & r & r \end{array}\right]$$

(3 points). 3 points for also recognising that division by zero is not permitted (i.e. dividing by $p + q = 0$ or $p = 0$) and this results in a different RREF.

5. (20 points) Answer the following True/False questions.

a. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices such that $\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}\mathbf{C}^{\mathrm{T}}$, then $\mathbf{M}^{\mathrm{T}} = \mathbf{C}\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{-1}$.

b. If a solution to the system of equations $\mathbf{Ax} = \mathbf{b}$ exists, this implies that the inverse of $\mathbf{A}$ exists.

c. The RREF of a matrix $\mathbf{A}$ is unique.

d. The set of all 2x2 matrices with the diagonal elements equal to 1 is a vector space.

e. If $\det(\mathbf{A}) \neq 0$ then $\mathbf{A}^{-1}$ exists.

SOLUTION

a. True. This follows from the multiplication rules for matrices.

b. False. There can be an infinite number of solutions and in this case there is no inverse.

c. True.

d. False. There is no additive identity.

e. True.