
INSTRUCTIONS:

- Computers, calculators, books, notes, and crib sheets are not permitted.
- Write your name, instructor's name, and recitation number on the front of your bluebook.
- Work all **five problems**. Start each problem on a new page.
- Show your work and clearly identify your final answer.

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1. (20 points) For the following equation

$$y'' - 4y' + 4y = \frac{e^{2t}}{t}$$

a. Find the characteristic equation. (5 points)
b. Find two linearly independent solutions to the homogeneous equation. (4 points)
c. Find the particular solution. (6 points)
d. Give the general solution. (2 points)
e. Solve the initial value problem for $t_0 = -1$, $y(t_0) = e^{-2}$, and $y'(t_0) = e^{-2}$. (3 points)

SOLUTION

a. Let $y(t) = e^{\lambda t}$. Substituting $y(t)$ into the homogeneous equation gives

$$y'' - 4y' + 4y = e^{\lambda t}\big(\lambda^2 - 4\lambda + 4\big) = 0.$$

Since, $e^{\lambda t} \neq 0$

$$\lambda^2 - 4\lambda + 4 = 0.$$

This is characteristic equation.

b. Find the eigenvalues (characteristic roots) using the characteristic equation. In this case

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

so we have a single eigenvalue, $\lambda = 2$. This gives one solution to the homogeneous equation,

$$y_1(t) = e^{2t}.$$

A second (linearly independent) solution can be obtained by multiplying the first solution by t,

$$y_2(t) = te^{2t}.$$

c. The particular solution can be found using the variations of parameters method. Assuming that the particular solution will be of the form $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, we can obtain the following expressions for v_1' and v_2'

$$\begin{aligned} v_1' &= \frac{-y_2(t)e^{2t}}{tW[y_1, y_2]}, \\ v_2' &= \frac{y_1(t)e^{2t}}{tW[y_1, y_2]}, \end{aligned}$$

where $W[y_1, y_2]$ is the Wronskian of y_1 and y_2 obtained in part b. The Wronskian is $W[y_1, y_2] = e^{4t}$ giving

$$\begin{aligned} v_1' &= -1 \Rightarrow v_1(t) = -t + c_1, \\ v_2' &= \frac{1}{t} \Rightarrow v_2(t) = \log|t| + c_2, \end{aligned}$$

where c_1 and c_2 are constants of integration. Ignoring terms that appear in the homogeneous equation gives a particular solution of

$$y_p(t) = t\,\log|t|\,e^{2t}.$$

d. The general solution is

$$y(t) = A_1e^{2t} + A_2te^{2t} + t\,\log|t|\,e^{2t}.$$

e. First, $y'(t)$ is

$$y'(t) = 2A_1e^{2t} + A_2(2t+1)e^{2t} + (2t+1)\log|t|e^{2t} + e^t.$$

Using the initial conditions gives the following system of equations

$$\begin{aligned} y(-1) &= e^{-2} = e^{-2}\big(A_1 - A_2\big), \\ y'(-1) &= e^{-2} = e^{-2}\big(2A_1 - A_2 + 1\big). \end{aligned}$$

Therefore, $A_1 = -1$, $A_2 = -2$, and the solution is

$$y(t) = e^{2t}\big(t\log|t| - t - 1\big).$$

2. (20 points)

a. Solve the homogeneous differential equation $y'' - 2y' + y = 0$. (6 points)

b. Find the solution to the above problem satisfying the initial value: $y(0) = 2, y'(0) = 5$. (4 points)

c. Predict the most suitable form of y_p for the nonhomogeneous differential equation $y'' - 2y' + y = te^t$ (*DO NOT SOLVE*). (4 points)

d. Find the particular solution to the equation in part c. (6 points)

SOLUTION

a. Find the characteristic equation. Let $y(t) = e^{\lambda t}$. Substitute $y(t)$ into the differential equation to get

$$e^{\lambda t}\left(\lambda^2 - 2\lambda + 1\right) = 0.$$

Therefore the characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ which has the repeated eigenvalue, $\lambda = 1$. Therefore, two linearly independent solution to the equation are

$$\begin{aligned} y_1(t) &= e^t, \\ y_2(t) &= t\,e^t. \end{aligned}$$

So the general solution is

$$y(t) = A_1\,e^t + A_2\,t\,e^t.$$

b. First find $y'(t)$,

$$y'(t) = A_1\,e^t + A_2\,(t+1)\,e^t.$$

Now obtain two equations using $y(0)$ and $y'(0)$

$$\begin{aligned} y(0) &= 2 = A_1, \\ y'(0) &= 5 = A_1 + A_2. \end{aligned}$$

This gives $A_1 = 2$ and $A_2 = 3$. So the solution to the initial value problem is

$$y(t) = 2\,e^t + 3\,t\,e^t.$$

c. The appropriate form of the particular solution is

$$y_p(t) = (A_0 + A_1\,t)\,t^2\,e^t.$$

d. Using the a solution of the form $y_p(t) = (A_0 + A_1\,t)\,t^2\,e^t$ gives

$$\begin{aligned} y_p' &= \left(2A_0 + (A_0 + 3A_1)\,t + A_1\,t^2\right)t\,e^t, \\ y_p'' &= \left(2A_0 + 2(2A_0 + 3A_1)t + (A_0 + 6A_1)t^2 + A_1 t^3\right)e^t. \end{aligned}$$

Substituting these expressions into the original equation leads to

$$2(A_0 + 3A_1 t)e^t = te^t.$$

Comparing terms, $A_0 = 0$ and $A_1 = 1/6$. Therefore the particular solution is

$$y_p(t) = \frac{1}{6}t^2e^t.$$

3. For the system of linear differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

a. Find all the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. (8 points)

b. Find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. (6 points)

c. Find the solution to the above problem satisfying the initial value: $x_1(0) = 4, x_2(0) = -2$. (6 points)

SOLUTION

a. From $|\mathbf{A} - \lambda\mathbf{I}| = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$, we get $\lambda_1 = \lambda_2 = 2$. For $\lambda = 2$, the matrix

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Its RREF is:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

The equation for the eigenvector $\mathbf{v}$ is $v_1 + v_2 = 0$. We derive from here that

$$\mathbf{v} = r \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ such that } r \in \mathbb{R} \setminus 0.$$

b. To find the general solution we need to solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$ which leads to $u_1 + u_2 = 1$ from which we get:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $\mathbf{x}_1 = e^{2t}\mathbf{v}$, $\mathbf{x}_2 = te^{2t}\mathbf{v} + e^{2t}\mathbf{u}$, The general solution is then given by $\mathbf{x} = c_1\,\mathbf{x}_1 + c_2\,\mathbf{x}_2$,

$$\mathbf{x} = \begin{pmatrix} c_2te^{2t} + (c_1 + c_2)e^{2t} \\ -c_2te^{2t} - c_1e^{2t} \end{pmatrix}.$$

c. Plug in the initial condition, we obtain $c_1 + c_2 = 4$ and $-c_1 = -2$. Solving these equations, we derive: $c_1 = c_2 = 2$ and the solution to the intial value problem is

$$\mathbf{x} = \begin{pmatrix} 2(t+2)e^{2t} \\ -2(t+1)e^{2t} \end{pmatrix}.$$

4. (20 points) Given the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

answer the following questions.

a. Find the eigenvalues of **A** (5 points).

b. Find the eigenvectors of **A** (10 points).

c. Give the eigenspace associated with each eigenvalue (5 points).

SOLUTION

a. Find the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\ &= (\lambda-3)^2(1-\lambda) \\ &= 0. \end{aligned}$$

Therefore, $\lambda_{1,2} = 3$ and $\lambda_3 = 1$.

b. Solve $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = 0$. Begin with the augmented matrix

$$\left[\begin{array}{ccc|c} 3-3 & 1 & 0 & 0 \\ 0 & 3-3 & 1 & 0 \\ 0 & 0 & 1-3 & 0 \end{array}\right]$$

$$\text{RREF} \quad \rightarrow \quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore,

$$\mathbf{v}_1 = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \text{ such that } r \in \mathbb{R} \setminus 0.$$

This is also the eigenvector associated with $\lambda_2 = 3$.

Solve $(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{v}_1 = 0$. Begin with the augmented matrix

$$\left[\begin{array}{ccc|c} 3-1 & 1 & 0 & 0 \\ 0 & 3-1 & 1 & 0 \\ 0 & 0 & 1-1 & 0 \end{array}\right]$$

$$\text{RREF} \quad \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore,

$$\begin{aligned} v_1 - 1/4 v_3 &= 0, \\ v_2 + 1/2 v_3 &= 0. \end{aligned}$$

Set $v_3 = 1$, thus

$$\mathbf{v}_3 = \begin{bmatrix} 1/4 \\ -1/2 \\ 1 \end{bmatrix}.$$

c.

$$\mathbb{E}_{\lambda_1} = \mathbb{E}_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

$$\mathbb{E}_{\lambda_3} = \text{span}\left\{ \begin{bmatrix} 1/4 \\ -1/2 \\ 1 \end{bmatrix} \right\}.$$

5. (20 points) Answer the following true/false questions:

a. If y_1 and y_2 are two solutions of $y'' - y' \sin(t) + y = 0$ then $y = c_1 y_1 + c_2 y_2$ is the general solution.

b. The vectors $\mathbf{u} = (1, 1, 1, 2)$, $\mathbf{v} = (0, 1, 2, 3)$, and $\mathbf{w} = (1, 0, 0, 0)$ form a basis of $\mathbb{R}^4$.

c. The equation $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a scalar multiple of a column in $\mathbf{A}$.

d. The solution space of $y'' + y' + y = \sin(t)$ is a 2-dimensional vector space.

e. If $\mathbf{Ax} = \mathbf{x}$, then $\mathbf{x}$ is an eigenvector of $\mathbf{A}$.

SOLUTION

a. F. y_1 and y_2 may be different but linearly dependent.

b. F. These vectors span a 3-dimensional space within $\mathbb{R}^4$

c. F. $\mathbf{b}$ must be a linear combination of the columns of $\mathbf{A}$, not just one of the columns.

d. F. The solution space is not a vector space.

e. F. $\mathbf{x} = \mathbf{0}$ satisfies the equation but it is not an eigenvector.