APPM 2360

Exam 1: February 2, 2005

ON THE FRONT OF YOUR BLUEBOOK write (1) your name, (2) the name of your lecture professor and time of your lecture, (3) the name of your recitation TA and the number of your recitation, and (4) a five-problem grading grid. WHEN YOU ARE FINISHED, remember to sign the honor code pledge on the front of your bluebook. WHEN YOU TURN IN YOUR EXAM, you must show your CU ID.

Show ALL of your work in your bluebook, and **box in your final answers.** A correct answer with no relevant work shown will receive no credit. Start each problem on **the top of a new page.** This exam is closed-book and no calculators are allowed. You are allowed a one-page crib sheet. **Other than your crib sheet, no other papers, books, snacks, or devices of any sort may be on your desk.**

Each problem is worth 25 points, for a total of 100 points.

1. (a) The ODE $y' + p(t)y = t\,e^t$ has the homogeneous solution $y_h(t) = ct$ (where c is an arbitrary constant). Find the function $p(t)$ and then find the general solution of this ODE using the Euler-Lagrange method (Variation of Parameters).

   (b) Solve the IVP $y' = \frac{y^2}{1+t^2}$, $y(0) = \frac{1}{2}$.

2. Consider the two ODEs

$$y' = y - t \qquad (1)$$

$$y' = (y-t)^2 \qquad (2)$$

   (a) Fully classify both ODEs: state which variable is independent and which is dependent, whether the ODEs are linear, separable and, if applicable, homogeneous.

   (b) Consider equation (2) with initial condition $y(0) = 1$. Use Euler's Method, with a stepsize of $h = 1/2$, to find an approximate solution to this IVP at $t = 1$.

   (c) Show that $y_p(t) = t + 1$ is a solution of both ODEs, with initial condition $y(0) = 1$.

   (d) Does either IVP have any other solution(s)? Justify your answer.

   (e) You may assume the following fact to be true:

$$y_1(t) = e^t \text{ is a solution to the ODE } y' = y$$

   Does it therefore follow that $y(t) = Cy_1 + y_p$ (where C is an arbitrary constant, and y_p is defined in (c)) is the general solution of equation (1)? Why/why not?

   (f) You may assume the following fact to be true:

$$y_2(t) = \frac{-1}{t} \text{ is a solution to the ODE } y' = y^2$$

   Does it therefore follow that $y(t) = Cy_2 + y_p$ (where C is an arbitrary constant, and y_p is defined in (c)) is the general solution of equation (2)? Why/why not?

3. A 2 gallon tank is initially full of pure water. Salt water of concentration of 3 lb/gal flows in at a rate given by the function $f(t)$ (*i.e.* the flow rate changes with time). At the same time, the mixed solution is drained from the tank at the same rate ($f(t)$). This system is modeled with the IVP:

$$\begin{aligned} s' &= 3f(t) - \frac{s}{2}f(t) \\ s(0) &= 0 \end{aligned}$$

(a) In the case that $f(t) = \frac{1}{1+t}$ gal/min solve for $s(t)$ using the method of Integrating Factors. What happens to the concentration of salt in the tank as $t \to \infty$? Does this make physical sense?

(b) Solve the ODE (you do not need to worry about the initial condition) for $s(t)$ in the case that $f(t) = \frac{1}{(1+t)^2}$ gal/min. What happens to the concentration of salt in the tank as $t \to \infty$? Does it depend on the initial condition?

(c) Sketch the two flow rates $f(t)$ used in (a) and (b), and determine the total volume of salt water that has been pumped through the system as $t \to \infty$ (for both flow rates). Use this result to explain the different results from (a) and (b).

4. Match the following ODEs with the corresponding direction field. For this question, you do NOT need to show your working/reasoning.

(i) $y' = (y-1)(y-3)^2$

(ii) $y' = (y-1)^2(y-3)$

(iii) $y' = ty - \sin(t)$

(iv) $y' = t(1-2y)$

(v) $y' = yt^3 - yt$