APPM 2360

Final Exam: May 2, 2005

1. (25 points)

   (a) Find the general solution of $y'' = 2y' + 3y$

   (b) Convert the equation in (a) to a first-order system of equations and write the system in matrix-vector form

   (c) Find the eigenvalues of the matrix in (b). How do they relate to your answer to (a)?

   (d) Find the general solution to the system in (b). How does this solution — in particular, the individual components of the vector solution — relate to your answer to (a)?

Solutions:

(a) $y'' - 2y' - 3y = 0$ so try $y = e^{rt}$ to get char. polynomial: $r^2 - 2r - 3 = (r-3)(r+1) = 0 \Rightarrow r = 3, -1$. Hence general solution is $y(t) = c_1 e^{3t} + c_2 e^{-t}$.

(b) Define $v = y'$, so $y' = v$ and $v' = y'' = 2y' + 3y = 3y + 2v$. The system is

$$\begin{aligned} y' &= v \\ v' &= 3y + 2v \end{aligned} \qquad \Leftrightarrow \qquad \begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$$

(c) $\det(A - \lambda I) = \lambda^2 - (0+2)\lambda + (0-3) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda = 3, -1$, which are the same are the roots of the char. poly. in (a).

(d) Finding the eigenvectors: $\lambda = 3 \Rightarrow (A - \lambda I) = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $\lambda = -1 \Rightarrow$ $(A - \lambda I) = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence the general solution is

$$\begin{pmatrix} y \\ v \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The first component of the vector solution is $y(t) = c_1 e^{3t} + c_2 e^{-t}$, which is the same as found in (a). The second component is $v(t) = 3c_1 e^{3t} - c_2 e^{-t}$, which is the derivative of $y(t)$ (as it should be, since v was defined as y').

2. (20 points) Find the general solution of $y'' - 4y' + 4y = f(x)$, where

   (a) $f(x) = 3e^{-x}$

   (b) $f(x) = 2\sin(2x)$

Solutions:

(a) Taking the homogeneous part: $y'' - 4y + 4y =$, try $y = e^{rx}$ and get char. polynomial: $r^2 - 4r + 4 = (r-2)^2 = 0 \Rightarrow r = 2$ (repeated). Hence $y_h(t) = c_1 e^{2x} + c_2 t e^{2x}$. Use Undetermined Coefficients to find a particular solution: $y_p = Ae^{-x} \Rightarrow y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Hence $y_p'' - 4y_p' + 4y_p = e^{-x}(A + 4A + 4A) = 9Ae^{-x} = 3e^{-x} \Rightarrow A = \frac{1}{3}$. Therefore, the general solution is $y(x) = c_1 e^{2x} + c_2 t e^{2x} + \frac{1}{3} e^{-x}$.

(b) The homogeneous part is the same as in (a). Again, using Undetermined Coefficients: $y_p = A\cos(2x) + B\sin(2x) \Rightarrow y_p' = -2A\sin(2x) + 2B\cos(2x) \Rightarrow y_p'' = -4A\cos(2x) - 4B\sin(2x)$. Hence, $y_p'' - 4y_p' + 4y_p = (-4A - 8B + 4A)\cos(2x) + (-4B + 8A + 4B)\sin(2x) = -8B\cos(2x) + 8A\sin(2x) = 2\sin(2x) \Rightarrow B = 0, A = \frac{1}{4}$. Therefore, the general solution is $y(x) = c_1 e^{2x} + c_2 t e^{2x} + \frac{1}{4}\cos(2x)$.

3. (30 points)

(a) Solve the IVP system

$$\begin{cases} x' &= -4x + 3y \\ y' &= -6x + 2y \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$$

(b) Find the general solution of

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

Solutions:

(a) Finding the eigenvalues of $\begin{pmatrix} -4 & 3 \\ -6 & 2 \end{pmatrix}$: $\lambda^2 - (-2)\lambda + (-8 - (-18)) = \lambda^2 + 2\lambda + 10 = 0 \Rightarrow$ $\lambda = \frac{1}{2}\left(-2 \pm \sqrt{4 - 40}\right) = -1 \pm 3i$. For $\lambda = -1 + 3i$: $A - \lambda I = \begin{pmatrix} -3 - 3i & 3 \\ -6 & 3 - 3i \end{pmatrix} \Rightarrow$ $-3(1 + i)v_1 + 3v_2 = 0 \Rightarrow (1 + i)v_1 = v_2 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\left[c_1\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(3t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(3t)\right) + c_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(3t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(3t)\right)\right]$$

Applying the initial condition:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $c_1 = 1 \Rightarrow c_2 = -1$. Hence the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(3t) - \begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin(3t)\right]$$

(b) Finding the eigenvalues of $\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$: $\lambda^2 - 0\lambda + (-1 - 3) = \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$. For $\lambda = 2$: $A - \lambda I = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. For $\lambda = -2$: $A - \lambda I = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \Rightarrow \mathbf{v} =$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

4. (20 points)

(a) Solve the system of equations

$$\begin{aligned} x + 2z &= -2 \\ 2x - y + z &= -7 \\ -x + 2y &= 8 \end{aligned}$$

(b) Find the determinant of the matrix that corresponds to this system.

Solutions:

(a)

$$\left(\begin{array}{rrr|r} 1 & 0 & 2 & -2 \\ 2 & -1 & 1 & -7 \\ -1 & 2 & 0 & 8 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 2 & -2 \\ 0 & -1 & -3 & -3 \\ 0 & 2 & 2 & 6 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

So the solution is $x = -2, y = 3, z = 0$.

(b) $\det \left(\begin{array}{rrr|r} 1 & 0 & 2 & -2 \\ 2 & -1 & 1 & -7 \\ -1 & 2 & 0 & 8 \end{array}\right) = 1 \left| \begin{array}{rr} -1 & 1 \\ 2 & 0 \end{array} \right| + 2 \left| \begin{array}{rr} 2 & -1 \\ -1 & 2 \end{array} \right| = (0-2)+2(4-1) = -2+6 = 4.$

5. (30 points) Answer the following TRUE/FALSE questions. You **MUST** write down an unabbreviated "**TRUE**" or "**FALSE**" to received credit. Note: In this problem you do not need to show your work.

(a) The general solution of $y' = (y-t)^2$ is $y = c(1+t)$, where c is a constant.

(b) In a 4-dimensional vector space, the span of any three vectors forms a 3-dimensional subspace.

(c) $y' - ty = t$ is a separable equation.

(d) A correct guess for the particular solution of $y'' - y = t\sin(t)$ is $y_p = c_1 t\cos(t) + c_2 t\sin(t)$.

(e) The functions $\{1, \sin^2(t), \cos^2(t)\}$ are linearly dependent.

(f) The eigenvalues of the matrices A and cA are the same for any number c.

(g) Any nonlinear, first order, autonomous equation is separable.

(h) The general solution of $y'' + \omega^2 y = \cos(\omega t)$ is an unbounded function.

(i) The rows of $A(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ are linearly <u>in</u>dependent, except for $\theta = 0, \pm\pi, \pm 2\pi, ...$

(j) $ty'' + y = t + 1$ <u>cannot</u> be solved using Undetermined Coefficients.

Solutions:

(a) FALSE

(b) FALSE

(c) TRUE

(d) FALSE

(e) TRUE

(f) FALSE

(g) TRUE

(h) TRUE

(i) FALSE

(j) TRUE

6. (25 points) Consider the I.V.P.

$$\begin{aligned} \frac{dy}{dt} &= \frac{y^2}{y^3 - 2ty} \\ y(0) &= 1\ . \end{aligned}$$

Solve this I.V.P. for $t(y)$ ($i.e.$ treat y as the independent variable) by noting that $\frac{dt}{dy} = 1/\left(\frac{dy}{dt}\right)$.

Solutions: $\frac{dt}{dy} = \frac{y^3-2ty}{y^2} = y - \frac{2}{y}t \Rightarrow \frac{dt}{dy} + \frac{2}{y}t = y$ which is a linear equation for $t(y)$. Using the method of Integrating Factors: $\mu = e^{\int 2/y dy} = e^{2\log|y|} = e^{\log|y^2|} = y^2$. Multiplying through by μ: $y^2\frac{dt}{dy} + 2yt = \frac{d}{dy}(y^2 t) = y^3$. Integrating both sides: $y^2 t = y^4/4 + C \Rightarrow t = \frac{y^2}{4} + \frac{C}{y^2}$. Applying the IC: $0 = \frac{1}{4} + \frac{C}{1} \Rightarrow C = -\frac{1}{4}$. Hence the solution is $t(y) = \frac{1}{4}(y^2 - \frac{1}{y^2})$.

7. (25 points) Consider the system of ODE's

$$\begin{aligned} x' &= \mu - y^2 \\ y' &= x - y \end{aligned}$$

(a) For $\mu = -2$, find all real equilibria. Classify each equilibrium by finding the eigenvalues of the linearized ODE's around each equilibria.

(b) Repeat for $\mu = 1$. Sketch the phase portrait of the (non-linear) system in this case.

(c) Define the bifurcation value, μ_0, as the value of μ where the number of equilibria changes. What is μ_0 for this system?

(d) Draw a bifurcation diagram for this system by plotting the y location of the equilibria as a function of μ. Be sure to label the stability of each equilibria on this plot.

Solutions:

(a) Setting $y' = 0$ gives $x = y$. Setting $x' = 0$ gives $-2 - y^2 = 0 \Rightarrow y^2 = -2$ which has no real solutions. So, no real equilibria.

(b) As in (a), we have $x = y$, but now $1 - y^2 = 0 \Rightarrow y = \pm 1$. So equilibria are $(1, 1)$ and $(-1, -1)$. The Jacobian is $DF = \begin{pmatrix} 0 & -2y \\ 1 & -1 \end{pmatrix}$. Hence $DF(1, 1) = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$ which has eigenvalues given by $\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = \frac{1}{2}(-1 \pm \sqrt{7}i)$, so $(1, 1)$ is a stable spiral. $DF(-1, -1) = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ which has eigenvalues given by $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0 \Rightarrow \lambda = -2, 1$. So $(-1, -1)$ is a saddle.

The eigenvectors for $DF(-1, -1)$ are $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (needed to sketch phase portrait).

(c) For a general μ, the equilibria are given by $\mu - y^2 = 0 \Rightarrow y = \pm\sqrt{\mu}$ which has real solutions only if $\mu \geq 0$. The change occurs at $\mu_0 = 0$ ($\mu < 0$ gives no equilibria, $\mu > 0$ gives two).

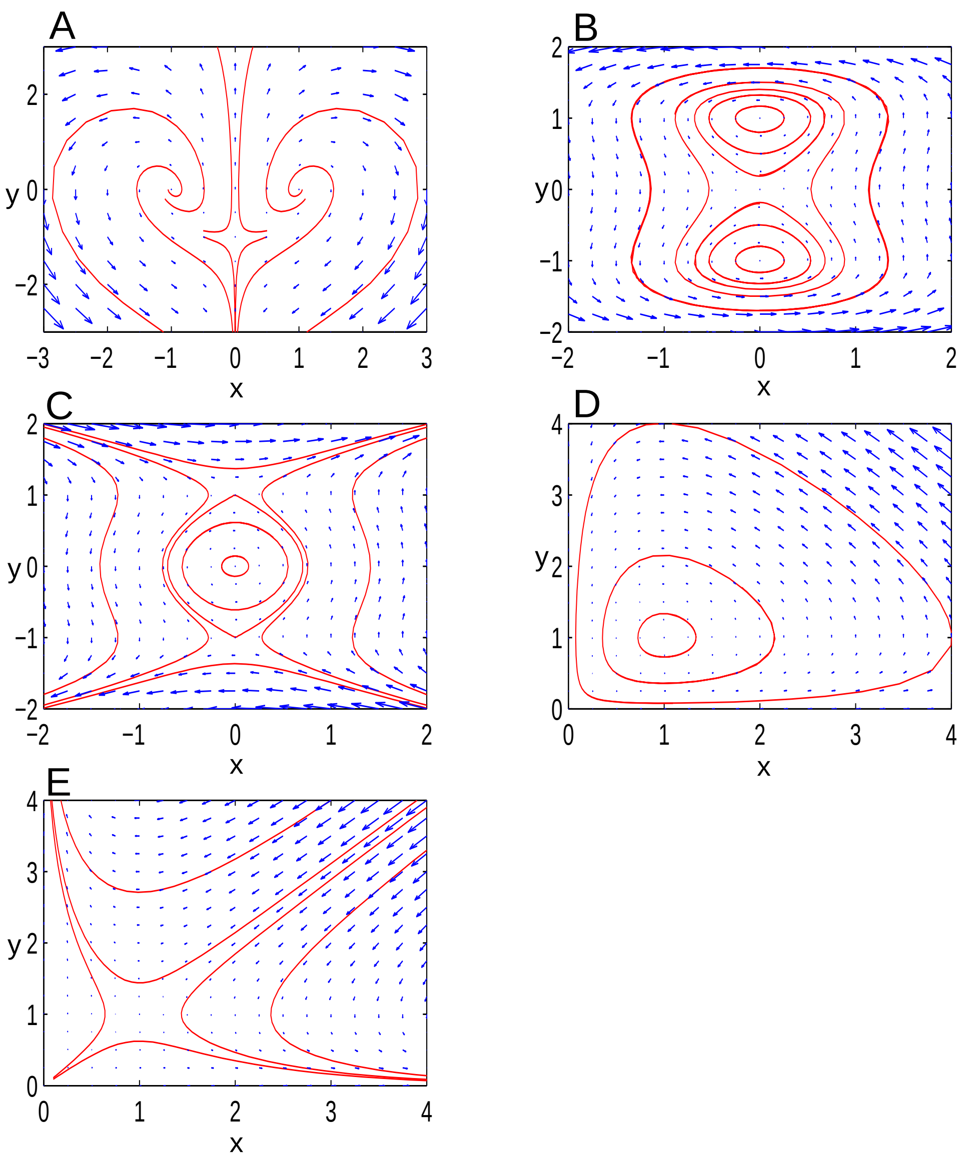
(d) As found in (b), the equilibrium at $(\sqrt{\mu}, \sqrt{\mu})$ is stable, while the equilibrium at $(-\sqrt{\mu}, -\sqrt{\mu})$ is unstable.

8. (25 points) Match the following systems with the correct phase portrait/vector field. Note: In this problem you do not need to show your work.

(i) $\begin{cases} x' = x - xy \\ y' = -y + xy \end{cases}$ (iv) $\begin{cases} x' = y - y^3 \\ y' = x \end{cases}$

(ii) $\begin{cases} x' = x - xy \\ y' = y - xy \end{cases}$ (v) $\begin{cases} x' = -y + y^3 \\ y' = x \end{cases}$

(iii) $\begin{cases} x' = xy \\ y' = y - x^2 + 1 \end{cases}$



Solutions: (i)-D (ii)-E (iii)-A (iv)-B (v)-C