INSTRUCTIONS:

- Computers, calculators, books and notes are not permitted.
- An 8.5x11 inch crib sheet is allowed.
- Write your name and instructor's name on the front of the test.
- Show your work and clearly identify your final answer.

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1. (15 points)

   For the following IVP

$$\frac{dy}{dt} = 3y^{2/3} \qquad y(2) = 0,$$

   (a) verify that $y_1(t) = 0$ is a solution.

   (b) Verify that $y_2(t) = (t-2)^3$ is also a solution.

   (c) Does this violate Picard's Theorem? Explain.

1. **Solution**

   (a) At first we check if $y_1(t) = 0$ satisfies the initial condition $y(2) = 0$. Since $y_1(2) = 0$ then the initial condition is satisfied. Second, we substitute solution y_1 into the differential equation to check if it satisfies the equation.

$$y_1(t) = 0 \Longrightarrow y_1'(t) = 0$$
$$\overset{\text{ODE}}{\Longrightarrow} y_1' = 3y_1^{2/3} \Rightarrow 0 = 3 \cdot 0.$$

   (b) Checking the initial condition we $y_2(2) = (2-2)^3 = 0$, which is satisfied. Substituting into the ODE,

$$\begin{aligned} y_2' = 3y_2^{2/3} \Rightarrow 3(t-2)^2 &= 3(t-2)^{3\frac{2}{3}} \\ &= 3(t-2)^2, \end{aligned}$$

   we see that the equation is satisfied.

   (c) No. Pricard's Theorem can not insure uniqueness of this IVP because $\frac{\partial f}{\partial y} = \frac{2}{3}y^{-2/3}$ does not exist at the point $(2,0)$ which is the initial condition $y(2) = 0$. Thus, the uniqueness theorem can not be used.

2. (30 points)

(a) Solve the following differential equation

$$y' = -\cos(t)\left(y^2 - 2y\right).$$

(b) Solve the following IVP

$$y' + \frac{1}{t+1}y = 2e^{3t}, \qquad y(0) = 1.$$

2. **Solution**

(a) Solving using separation of variables and partial fractions.

$$y' = -\cos(t)\left(y^2 - 2y\right) \Rightarrow \frac{dy}{y\,(y-2)} = -\cos(t)dt. \tag{1}$$

Partial fractions:

$$\begin{aligned}
\frac{1}{y\,(y-2)} = \frac{A}{y} + \frac{B}{y-2} &= \frac{A(y-2)}{y(y-2)} + \frac{By}{y(y-2)}, \\
\text{equality of numerators} &\Rightarrow 1 = A(y-2) + By, \\
&\Rightarrow 1 = -2A, \quad 0 = A + B, \\
&\Rightarrow A = -\frac{1}{2}, \quad B = \frac{1}{2}.
\end{aligned}$$

Thus, the we can rewrite (1) as

$$\frac{-dy}{2y} + \frac{dy}{2(y-2)} = -\cos(t)dt.$$

Integrating,

$$\begin{aligned}
\int \frac{-dy}{2y} + \int \frac{dy}{2(y-2)} &= \int -\cos(t)dt, \\
\Rightarrow -\frac{1}{2}\ln(|y|) + \frac{1}{2}\ln(|y+2|) &= -\sin(t) + C, \\
\Rightarrow \frac{1}{2}\ln\left(\frac{|y-2|}{|y|}\right) &= -\sin(t) + C, \\
\Rightarrow \ln\left(\frac{|y-2|}{|y|}\right) &= -2\sin(t) + C, \\
\overset{y>2}{\Longrightarrow} \frac{y-2}{y} &= Ce^{-2\sin(t)}, \\
\Rightarrow \left(1 - Ce^{-2\sin(t)}\right) y &= 2, \\
\Rightarrow \boxed{y(t) = \frac{2}{1 + Ce^{-2\sin(t)}}}.
\end{aligned}$$

Another way: using the Bernoulli formalism. We can write the ODE as

$$y' - 2\cos(t)y = -\cos(t)y^2,$$

which is the Bernoulli equation with $\alpha = 2$. If we let $v = y^{1-\alpha}$, we get

$$v = y^{-1},$$

which then we can substitute into the ODE by

$$\begin{aligned} y = v^{-1} &\Rightarrow y' = \frac{-1}{v^2}v', \\ &\overset{\text{ODE}}{\Longrightarrow} \frac{-1}{v^2}v' + 2\cos(t)\frac{1}{v} = -\cos(t)\frac{1}{v^2}, \\ \text{multiply by } -v^2 &\Rightarrow \boxed{v' + 2\cos(t)v = \cos(t)}. \end{aligned} \tag{2}$$

This is a linear ODE in the variable v, which we can solve using integrating factor. The integrating factor is

$$\mu = e^{\int 2\cos(t)dt} = e^{2\sin(t)},$$

so that multiplying both sides of (2) by μ results in

$$\frac{d}{dt}\left[e^{2\sin(t)}v\right] = e^{2\sin(t)}\cos(t). \tag{3}$$

Integrating the $r.h.s$ and using $u = 2\sin(t)$ in our u-substitution, we get

$$\begin{aligned} \int e^{2\sin(t)}\cos(t)\ dt &= \int \frac{1}{2}e^u\ du = \frac{1}{2}e^u + C, \\ &= \frac{1}{2}e^{2\sin(t)} + C. \end{aligned}$$

Thus, integration of (3) is

$$e^{2\sin(t)}v(t) = \frac{1}{2}e^{2\sin(t)} + C.$$

Thus,

$$v(t) = \frac{1}{2} + Ce^{-2\sin(t)}. \tag{4}$$

Substituting for $v(t) = 1/y(t)$ in (4), taking the reciprocal and multiplying by $\frac{2}{2}$, we get

$$\boxed{y(t) = \frac{2}{1 + Ce^{-2\sin(t)}}}.$$

(b) Using the integrating factor method,

$$\mu = e^{\int \frac{1}{t+1}\ dt} = e^{\ln(|t+1|)} = t + 1 \text{ for } t > -1.$$

Then multiplying both sides of the ODE by μ we have

$$\frac{d}{dt}\left[(t+1)y\right] = 2(t+1)e^{3t},$$

which when integrated (using *integration by parts*) becomes

$$\begin{aligned} (t+1)y &= 2\int (t+1)e^{3t}dt \underset{dv = e^{3t}dt}{\overset{u = t+1}{\Longrightarrow}} 2\left[(t+1)\frac{e^{3t}}{3} - \int \frac{e^{3t}}{3}\ dt\right] \\ &= 2\left[(t+1)\frac{e^{3t}}{3} - \frac{e^{3t}}{9}\right] + C. \end{aligned}$$

Thus,

$$y(t) = 2\left[\frac{e^{3t}}{3} - \frac{e^{3t}}{9(t+1)}\right] + \frac{C}{t+1}.$$

Using the intial condition $y(0) = 1$, we get

$$\begin{aligned} y(0) &= 2\left[\frac{1}{3} - \frac{1}{9}\right] + C, \\ \Rightarrow 1 &= \frac{4}{9} + C, \\ \Rightarrow C &= \frac{5}{9}. \end{aligned}$$

So the answer to the IVP is

$$\boxed{y(t) = 2\left[\frac{e^{3t}}{3} - \frac{e^{3t}}{9(t+1)}\right] + \frac{5}{9(t+1)}},$$

or if the $r.h.s$ of the ODE is multiplied out, the result may look like

$$\boxed{y(t) = \frac{2\left[\frac{1}{3}te^{3t} - \frac{2}{9}e^{3t}\right]}{t+1} + \frac{5}{9}\frac{1}{t+1}}.$$

3. (27 points)

Our job is to dilute 100 gals. of salt water in a tank. Fresh water is poured at a rate of 2 gal/min into the tank that has an initial concentration of 1 lb/gal. The stirred mixture is drained out of the tank at a rate of 2 gal/min.

(a) Find the amount of salt in the tank at anytime.

(b) Suppose the outflow drain ruptures at 50 mins, doubling the outflow rate. Find the amount of salt in the tank since the rupture.

(c) How long will it take after the rupture for the tank to be empty?

3. **Solution**

(a) Since the concentration of water being poured into the tank is zero (fresh water), the ODE that models the amount of salt in the tank is

$$\frac{dx}{dt} = 2 \cdot 0 - \frac{x\text{lbs}}{100\text{gals}} \cdot 2\,[\text{gal/min}]\,,$$

with an initial condition of

$$\begin{aligned} x(0) &= \text{intial concentration} \times \text{intial amout of salt water in tank,} \\ &= 1\,[\text{lbs/gal}] \times 100\,[\text{gal}]\,, \\ x(0) &= 100\,[\text{lbs}]\,. \end{aligned}$$

So solving

$$x' = \frac{-x}{50}$$

for x(t)

$$x(t) = Ce^{\frac{-t}{50}},$$

and using the initial condition, we have

$$\boxed{x(t) = 100e^{\frac{-t}{50}}\,[\text{lbs}]}.$$

(b) First we must find out how much salt is in the tank at $t = 50$ mins, which is

$$x(50) = 100e^{\frac{-50}{50}} = 100e^{-1}\,[\text{lbs.}]$$

The "Rate In" is still zero since fresh water is still poured into the tank. The "Rate Out", is

$$\begin{aligned} \text{Rate Out} &= \frac{\text{salt in tank}}{\text{mixture in tank}} \cdot \text{flow rate out,} \\ &= \frac{x(t)}{100\,[\text{gal}] - 2\,[\text{net flow rate (in gal/min)}] \cdot t\,[\text{min}]} \cdot 4\,[\text{gal/min}]\,. \end{aligned}$$

So the IVP that models the salt in the tank after the rupture is

$$x' = -\frac{4x}{100 - 2t} \qquad x(0) = 100e^{-1}.$$

The initial time is not $t = 50$, but is $t = 0$ since we consider this model only valid from the time the pipe ruptured and for time into the future (until the tank drains).

The ODE is solved using seperation of variables

$$\begin{aligned}
\frac{dx}{x(t)} &= -\frac{4\ dt}{100-2t}, \\
\Rightarrow \ln(|x(t)|) &= -4\int \frac{-1}{2}\frac{du}{u}, \text{ where } u = 100-2t, \\
\Rightarrow \ln(|x(t)|) &= 2\ln(100-2t)+C, \\
\Rightarrow x(t) &= C(100-2t)^2.
\end{aligned}$$

Using the initial condition,

$$x(0) = C(100)^2 = 100e^{-1} \Rightarrow C = \frac{e^-1}{100},$$

we have

$$\boxed{x(t) = \frac{e^-1}{100}(100-2t)^2\ \text{[lbs]}}$$

of salt in the tank after the rupture and before it is drained.

(c) The tank is drained in time t that satisfies

$$100\,[\text{gal}] - 2\,[\text{net flow rate (in gal/min)}]\cdot t\,[\text{min}] = 0\,[\text{gal}]\,.$$

The solution is

$$\boxed{t = 50\text{min}}.$$

4. (18 points)

For the following $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are $n \times n$ matricies. Are the following statements necessarily true, *yes* or *no*? Give a short reason why or why not.

(a) The linear system $\mathbf{Ax} = \mathbf{0}$ has the uniqe solution $\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ and $\mathbf{0}$ are $n \times 1$ matricies.

(b) If the RREF of an augmented matrix has a row of the form $[0\ 0\ \cdots\ 0|k]$ then the associated system of equations is inconsitent.

(c) If $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$.

(d) If $\mathbf{A}$ is invertible then $\left(\mathbf{ABA}^{-1}\right)^3 = \mathbf{AB}^3\mathbf{A}^{-1}$.

(e) If $|\mathbf{A}||\mathbf{B}| = 0$ then $(\mathbf{AB})^{-1}$ does not exist.

(f) If $|\mathbf{A}||\mathbf{B}| = |\mathbf{B}||\mathbf{A}|$ then $\mathbf{AB} = \mathbf{BA}$.

4. **Solution**

(a) No. $\mathbf{Ax} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$, but it isn't necessarily unique. Consider $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) No. If $k = 0$, then the system is consistent.

(c) No. We can not simply divide by a matrix. This would always be true if we knew $\mathbf{A}$ was invertible.

(d) Yes. For matrices, exponents do not distribute like they do for numbers, $\left(\mathbf{ABA}^{-1}\right)^3 = (\mathbf{ABA}^{-1})(\mathbf{ABA}^{-1})(\mathbf{ABA}^{-1}) = \mathbf{AB}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B}(\mathbf{A}^{-1}\mathbf{A})\mathbf{BA}^{-1} = \mathbf{ABBBA}^{-1} = \mathbf{AB}^3\mathbf{A}^{-1}$.

(e) Yes. A property of determinants is that $|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}|$. Since $0 = |\mathbf{A}||\mathbf{B}| = |\mathbf{AB}|$, $(\mathbf{AB})$ is not invertible.

(f) No. Matrix multiplication is *not* commutative! The fact that the determinants are equal has nothing to do with anything.

5. (10 points)

Find a value of k to make

$$\begin{aligned} x + 3y &= 2 \\ -2x + ky &= 1 \end{aligned}$$

an inconsistent system of equations.

5. **Solution** First, write the system in augmented matrix form.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & k & 1 \end{array}\right].$$

Row reduce by replacing *row 2* with *row 2 + 2×row 1*,

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & k+6 & 5 \end{array}\right].$$

From here, we see if $k + 6 = 0$, we have a row of all zeros augmented with a non-zero. Thus, if $k = -6$, the system is inconsistent.

6. (25 points)

For the following system

$$\begin{aligned}\frac{dx}{dt} &= 1 - x - y \\ \frac{dy}{dt} &= x - y^2 + 1,\end{aligned}$$

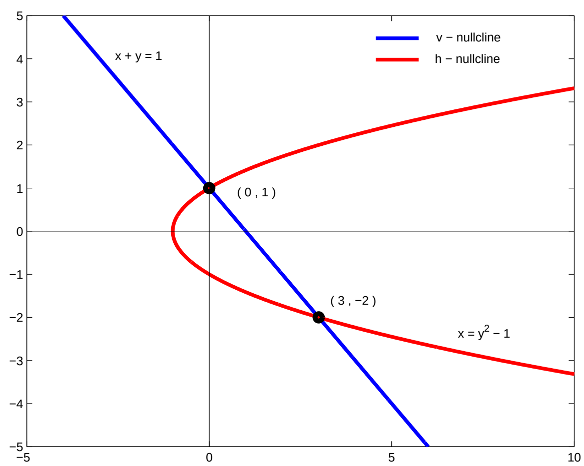
(a) Determine and plot the equilibrium points and nullclines of the system.

(b) What is the slope of the solution at the point $(1, 1)$?

6. **Solution** (a)

v-nullclines : $x' = 1 - x - y = 0$	**h-nullclines** : $y' = x - y^2 + 1 = 0$
$x + y = 1$	$x = y^2 - 1$

The equilibrium points are at the intersections of the two nullclines, $(0, 1)$ and $(3, -2)$.



(b) The slope of the solution is given by the differential equations. At $(x, y) = (1, 1)$, $dx/dt = 1 - 1 - 1 = -1$, and $dy/dt = 1 - 1^2 + 1 = 1$. So the slope is change in y over change in x, or $\frac{dy/dt}{dx/dt} = \frac{1}{-1} = -1$.

7. (25 points)

(a) Is the collection of all solutions to the following differential equation

$$y' + (e^t + 4)y - 2t + 6 = 0$$

a vector space? Explain.

(b) What is a basis of the subspace of $\mathbb{R}^4$ defined by the equation,

$$x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

Show that the basis vectors are linearly indepenedent.

7. **Solution**

(a) No, the set of the solutions to the given DE do not form a vector space. The set of solutions does not have a zero vector ($y(t) = 0$ is not a solution to the ODE), it is not closed under addition and it is not closed under scalar multiplication.

(This is sufficient for full credit on this problem)

To show, for example, it is not closed under addition, assume $u(t)$ and $v(t)$ are solutions, i.e. $u' + (e^t + 4)u - 2t + 6 = 0$ and $v' + (e^t + 4)v - 2t + 6 = 0$. Is $u(t) + v(t)$ also a solution?

$$\begin{aligned}(u+v)' + (e^t+4)*(u+v) - 2t + 6 &= u' + v' + (e^t+4)u + (e^t+4)v - 2t + 6 \\ &= \left[u' + (e^t+4)u - 2t + 6\right] + \left[v' + (e^t+4)v - 2t + 6\right] + 2t - 6 \\ &= 0 + 0 + 2t - 6 \\ &\neq 0,\end{aligned}$$

So $u(t) + v(t)$ is not a solution and the set of solutions is not closed under addition.

(b) Rewrite the equation in matrix-vector form,

$$\begin{bmatrix} 1 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

Since the "matrix" only has a pivot in the first column, we have three free variables, $x_2 = r$, $x_3 = s$, $x_4 = t$. This means $x_1 = x_2 - 2x_3 - 4x_4 = r - 2s - 4t$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r - 2s - 4t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the subspace of $\mathbb{R}^4$ defined by the equation (aka solution space) is given by the set of vectors,

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To check for linear independence, show the only solution to $c_1v_1 + c_2v_2 + c_3v_3 = 0$ is $c_1 = c_2 = c_3 = 0$. As an augmented matrix, this is

$$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The RREF of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is a pivot in every column of the matrix, there is a unique solution, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Thus, the three vectors are linearly independent.