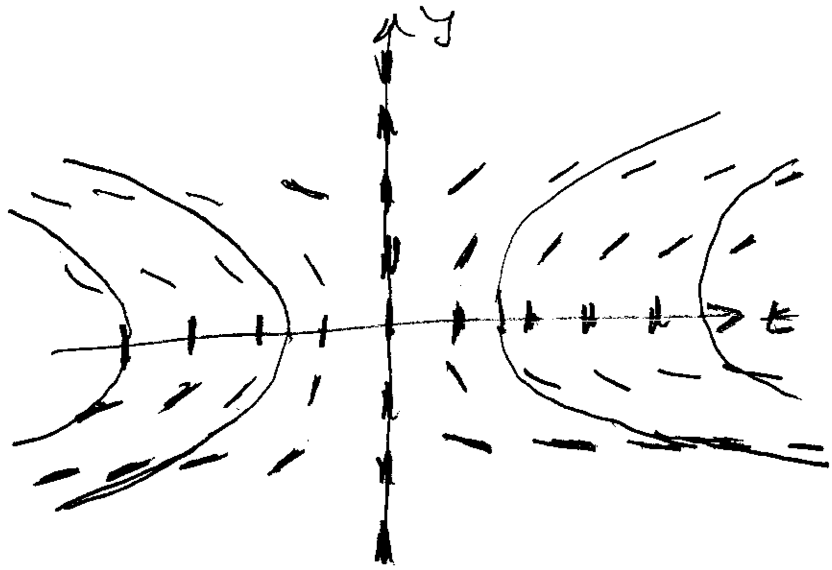
1 a, $2tyy' = 1 \Rightarrow y' = \underbrace{\frac{1}{2ty}}_{f(t,y)}$

The values of $f(t,y)$ are infinite when $y=0$ or $t=0$.
They are positive when $ty>0$ and negative when $ty<0$
They go to zero inversely proportionally to the value of $t \cdot y$.
The following is a sketch, with some possible solution curves drawn:



b, The DE is separable.
Write it as

$$2yy' = \frac{1}{t}$$

Integrate

$$y^2 = C + \ln|t|,$$

$$\Rightarrow \quad y = \pm\sqrt{C + \ln|t|}.$$

2. $y' = \frac{y}{t} + \frac{y^2}{t^2}$

a. Given the hint that no calculations are needed to find a solution satisfying $y(1)=0$, we note that $y(t)=0$ satisfies both equation and initial condition.

If we do not spot this, using that the equation is Euler homogeneous gives (after quite a lot of calculations)

$$y(t) = \frac{t}{c - \ln t}$$, forcing $c = \infty$ and again $y(t)=0$

b. Set $y = y_p + \frac{1}{v}$, choose $y_p = 0$.

$\Rightarrow y' = -\frac{1}{v^2} \cdot v'$, $v' + \frac{1}{t} v = -\frac{1}{t^2}$

For example, by integrating factor:

$IF = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$\frac{d}{dt}(t \cdot v) = -\frac{1}{t}$ $(t>0)$

$tv = -\ln t + c$, $y = \frac{t}{c - \ln t}$

$y(1) = 1 \Rightarrow c = 1$

So $y(t) = \frac{t}{1 - \ln t}$

c. Bernoulli: $y' + p(t) y = q(t) \cdot y^{\alpha}$

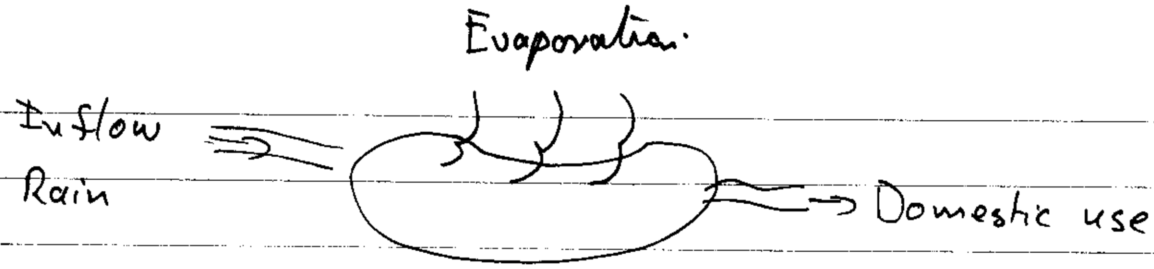
Here, $\alpha = 2$,

Transformation $v = y^{1-\alpha}$ becomes

$y = \frac{1}{v}$;

Same as in Riccati case (part b).

#3.



a. x - Total amount of salt.
Initially total amount of salt is: $\mu V(0)$
No salt inflow (fresh water)
Outflow concentration: $\frac{x(t)}{V(t)}$
Outflow rate (domestic use): ρ

Model: $$\frac{dx}{dt} = 0 - \frac{x}{V(t)}\rho, \quad x(0) = \mu V(0)$$

b. For constant volume: $V(t) = V_0$

$$\frac{dx}{dt} = -\frac{x}{V_0}\rho, \quad x(0) = \mu V_0$$

$$\int \frac{dx}{x} = -\int \frac{\rho}{V_0}\, dt + \text{const.}$$

$$\ln|x| = -\frac{\rho}{V_0} t + \text{const}$$

$$x(t) = \text{const}\; e^{-\rho/V_0\, t}$$

Since $x(0) = \mu V_0 = \text{const}$

$$x(t) = \mu V_0\, e^{-\frac{\rho}{V_0} t}$$

After $2005 - 1970 = 35$ years:

$$x(35) = \mu V_0\, e^{-\frac{\rho}{V_0} 35}$$

4. Rewrite to 'standard form'

$$y' - \frac{t}{1+t^2}y = \frac{1}{\sqrt{1+t^2}}$$

a, Homogeneous equation

$$y' - \frac{t}{1+t^2}y = 0$$

Separable:

$$\frac{y'}{y} = \frac{t}{1+t^2}, \quad \int \frac{1}{y}dy = \int \frac{t}{1+t^2}dt$$

$$\ln|y| = \frac{1}{2}\ln(1+t^2) + C$$

$$y_h = d \cdot \sqrt{1+t^2}$$ — Homogeneous solution.

b. For a particular solution, try

$$y_p = d(t)\sqrt{1+t^2}$$

Substituting into the ODE gives

$$\underbrace{d'\sqrt{1+t^2} + \cancel{\frac{t}{\sqrt{1+t^2}}d}}_{y'} - \cancel{\frac{t}{1+t^2}d\sqrt{1+t^2}} = \frac{1}{\sqrt{1+t^2}}$$

$$d' = \frac{1}{1+t^2}, \quad d = \tan^{-1}t$$. (need no constant)

c. The general solution is

$$y = y_h + y_p = d\sqrt{1+t^2} + \sqrt{1+t^2}\tan^{-1}t$$

IC $y(0) = 0 \Rightarrow d = 0$.

Then $y(t) = \sqrt{1+t^2}\tan^{-1}t$.

5. $y' = y^2 - \varepsilon$

a. Equilibrium solutions when $y^2 - \varepsilon = 0$,

$y = \pm\sqrt{\varepsilon}$

There are two solutions for $\varepsilon > 0$, one for $\varepsilon = 0$ and no for $\varepsilon < 0$.

Stability:

$f(y) = y^2 - \varepsilon$

$\varepsilon < 0 \quad f > 0 \Rightarrow y$ increasing

$\varepsilon = 0 \quad f \geq 0 \Rightarrow y$ increasing, but equilibr. at $y=0$

$\varepsilon > 0 \quad f < 0$ if $-\sqrt{\varepsilon} < y < \sqrt{\varepsilon}$, y decreasing

$\qquad\quad f > 0$ if $y < -\sqrt{\varepsilon}$ or $y > \sqrt{\varepsilon}$, y increasing.

So $y = +\sqrt{\varepsilon}$ unstable, $y = -\sqrt{\varepsilon}$ stable. } Easiest seen after plotted in bifurcation diagram

b.