APPM 2360–Exam 2

Solutions **October 19, 2005, 7–8.30pm**

1. (a) For

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

(b) Since $(AB)^T = B^T A^T$,

$$(AB)^T = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^T.$$

(c) For

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

The product BA is not defined since the number of columns of B need to equal the number of rows of A.

2. (a) Rewrite the system as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 4 \\ 1 & 1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right].$$

We therefore have

$$\begin{aligned} x + y + z &= 1 \\ -3y - z &= 2 \\ -2z &= -2 \end{aligned}$$

so that $z = 1, y = -1, x = 1$.

(b) Rewrite the system as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & \text{-1} & 1 & 0 \\ 1 & \text{-2} & 0 & \lambda \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & \text{-3} & \text{-1} & \text{-4} \\ 0 & \text{-3} & \text{-1} & \lambda-2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & \text{-3} & \text{-1} & \text{-4} \\ 0 & 0 & 0 & \lambda+2 \end{array}\right].$$

It has infinite number of solutions for $\lambda = -2$. For this value we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ \text{-4} \end{bmatrix}$$

so that $3y = 4 - z$; $x = 2 - y - z = 2 - (4 - z)/3 - z = \frac{2}{3} - \frac{2}{3}z$, or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} - \frac{2}{3}z \\ \frac{4}{3} - \frac{1}{3}z \end{bmatrix}$$

for any value of z.

3. (a) In this question you may calculate the determinant any way you want. The easiest is to use the row reductions of question 2, in which case the determinant is just the product of the diagonal elements. From question 2,

$$A \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \text{-3} & \text{-1} \\ 0 & 0 & 0 \end{bmatrix}, \; B \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \text{-3} & \text{-1} \\ 0 & 0 & \text{-2} \end{bmatrix}.$$

Therefore $\det(A) = 1 \times (-3) \times 0 = 0$, and $\det(B) = 1 \times (-3) \times (-2) = 6$.

(b) Since the third column has a free variable, the first two columns are independent and a basis is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \text{-1} \\ \text{-2} \end{bmatrix}.$$

Alternatively, you can show that the three columns are dependent but any two of them are independent, i.e. any two of the three columns also form a basis.

(c) i. Since $\det(kA) = k^n \det(A) \neq 1$ for any $k \neq 1$, these matrices do not form a vector space (multiples of the vector are not in the space).

ii. Again we only need to observe that $\text{trace}(kA) = nk \neq 1$ for any $k \neq \frac{1}{n}$ (multiples of the vector ore not in the space)..

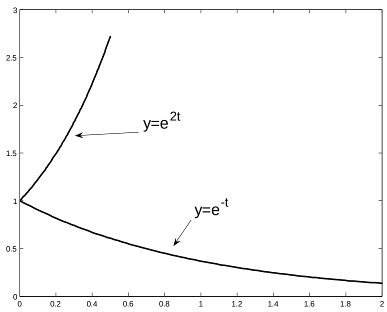

Figure 1:

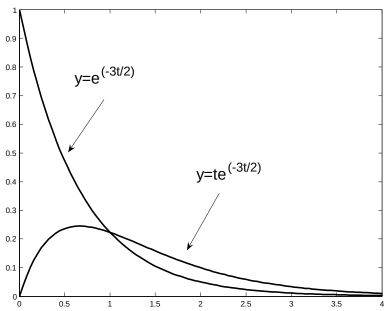

Figure 2:

4. (a) $y'' - y' - 2y = 0$. Try $y = e^{rt}$, then the characteristic equation is $r^2 - r - 2 = 0$, or $(r-2)(r+1) = 0$. We therefore have two solutions,

$$y = e^{2t} \text{ and } y = e^{-t}.$$

The general solution is therefore

$$y = Ae^{2t} + Be^{-t}$$

(See Figure 1)

(b) $4y'' + 12y' + 9y = 0$. Trying $y = e^{rt}$, the characteristic equation is $4r^2 + 12r + 9 = 0$, or $(2r+3)^2 = 0$. In this case the charactersitic equation gives us only one solution $y = e^{-\frac{3}{2}t}$. A second independent solution is $y = te^{-\frac{3}{2}t}$ and the general solution is

$$y = Ae^{-\frac{3}{2}t} + Bte^{-\frac{3}{2}t}$$

(See Figure 2).

5. Given

$$\begin{aligned} x' &= y \\ y' &= -y - x + x^3 \end{aligned}$$

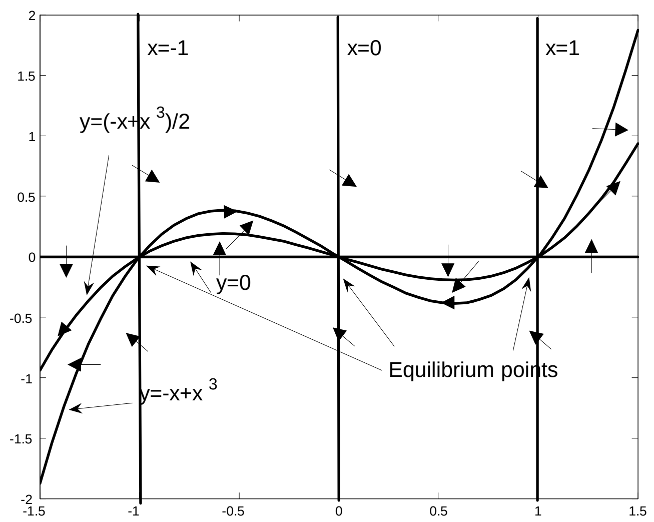

Figure 3:

(a) We start with the two nullclines (special cases of isoclines). (1) $x' = 0$, i.e. $y = 0$, and (2) $y' = 0$, i.e. $y = -x + x^3$.The general isoclines are given by

$$\frac{dy}{dx} = \frac{-y - x + x^3}{y} = c, \text{ a constant.}$$

The isoclines $\frac{dy}{dx} = -1$, are given by $\frac{-y-x+x^3}{y} = -1$, or $-x + x^3 = 0$, i.e. the lines $x = -1, x = 0, x = 1$. The isocline $\frac{dy}{dx} = 1$ is given by $\frac{-y-x+x^3}{y} = 1$, or $y = (-x + x^3)/2$. The general isocline $\frac{dy}{dx} = c$, is given by $y = (-x + x^3)/(1 + c)$, i.e. the curves $y = (-x + x^3)/C$.

(b) The equilibrium points are the interections of the nullclines, $y = 0$ and $-y - x + x^3 = 0$, i.e. the points $(0, 0)$, $(1, 0)$, and $(-1, 0)$. (See Figure 3)