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Appm 2360 - Exam 3 - Solutions

① $y'' + 2y' + 2y = \sin t$

a) Characteristic eqn: $r^2+2r+2=0$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\Rightarrow y_h(t) = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$$

$$= \underline{e^{-t}(k_1 \cos t + k_2 \sin t)}$$

b) Try: $y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$\therefore (-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) = \sin t$

$$\begin{aligned} A + 2B &= 0 \\ -2A + B &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= \frac{-2}{5} \\ B &= \frac{1}{5} \end{aligned}$$

$$y_p(t) = -\frac{2}{5}\cos t + \frac{1}{5}\sin t$$

c) $$y(t) = y_h + y_p = e^{-t}(k_1\cos t + k_2\sin t) - \frac{2}{5}\cos t + \frac{1}{5}\sin t$$

$$y(0) = k_1 - \frac{2}{5} = 0$$

$$\therefore k_1 = \frac{2}{5}$$

$$y'(t) = e^{-t}\left([-k_1+k_2]\cos t + [-k_2-k_1]\sin t\right) + \frac{2}{5}\sin t + \frac{1}{5}\cos t$$

$$y'(0) = -k_1 + k_2 + \frac{1}{5} = 0$$

$$\therefore k_2 = \frac{1}{5}$$

$$\Rightarrow y(t) = \frac{2}{5}(e^{-t}-1)\cos t + \frac{1}{5}(e^{-t}+1)\sin t$$

(2) $y'' + 2by' + 4y = 4$

(a) Associated homogeneous problem: $y'' + 2by' + 4y = 0$

Characteristic eqn: $r^2 + 2br + 4 = 0$

$$r_{1,2} = -b \pm \sqrt{b^2-4}$$

$\therefore$ for $b \neq 2$

$$y_h(t) = c_1 e^{(-b+\sqrt{b^2-4})t} + c_2 e^{(-b-\sqrt{b^2-4})t}$$

(b) With $b > 0$ we have damping $r_{1,2} < 0$ and the solution decays to zero as $t \to \infty$

(c) For $b = 2$, $r_{1,2} = -b = -2$ and we have a repeated real root,

$$y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

(d) Variation of parameters:

taking: $y_p = v_1(t) e^{-2t} + v_2(t) t e^{-2t}$

$$e^{-2t} v_1' + t e^{-2t} v_2' = 0$$

$$-2e^{-2t} v_1' + (1-2t) e^{-2t} v_2' = 4$$

$$\therefore \quad e^{-2t} v_2' = 4, \quad v_2' = 4e^{2t}$$

$$e^{-2t} v_1' = -4t, \quad v_1' = -4te^{2t}$$

$$v_2(t) = 2e^{2t} \qquad v_1(t) = -4\int te^{2t}\,dt$$

$$= -4\left[\frac{t}{2}e^{2t} - \int \frac{1}{2}e^{2t}\right]$$

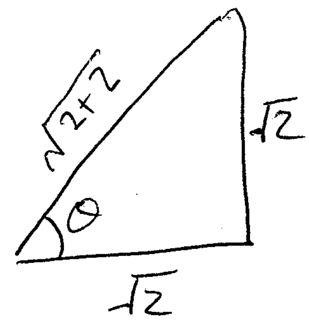
$$= -2te^{2t} + e^{2t}$$

$$\therefore \ \varphi_p(t) = [-2t+1] + 2t = 1$$

3

a) $\sqrt{2}+\sqrt{2}i = \sqrt{2+2}\, e^{i\frac{\pi}{4}}$

$= 2e^{i\frac{\pi}{4}}$



b) $e^{a+ib} = e^{a}(\cos b + i\sin b) = 3$

$\therefore\ a = \ln 3$

$b = 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

c) $\mathrm{Im}\left\{\frac{1}{2+2i}\right\} = \mathrm{Im}\left\{\frac{2-2i}{(2+2i)(2-2i)}\right\}$

$= \mathrm{Im}\left\{\frac{2-2i}{8}\right\}$

$= \frac{-1}{4}$

d) $\int e^{2x}\sin 2x\,dx = \mathrm{Im}\left\{\int e^{2x}e^{2ix}dx\right\}$

$\int e^{(2+2i)x}dx = \frac{1}{2+2i}e^{2x}e^{2ix} = \frac{2-2i}{8}e^{2x}e^{2ix}$

$$\operatorname{Im}\left\{\left(\frac{1}{4}-\frac{i}{4}\right)(\cos 2x + i\sin 2x)e^{2x}\right\} = e^{2x}\left(-\frac{1}{4}\cos 2x + \frac{1}{4}\sin 2x\right)$$

$$\int e^{2x}\sin 2x\,dx = \frac{1}{4}e^{2x}(\sin 2x - \cos 2x)$$

4 a) If $\det(A) = 0$ then $A\underline{u} = \lambda\underline{u}$ with $\lambda = 0$ for some $\underline{u} \neq \underline{0}$. We will have at least one eigenvalue of A being zero.

b)
$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 4-\lambda & 3 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(3-\lambda)-6\right] + \left(-(3-\lambda)-3\right) = 0$$

$$-\lambda^3 + 8\lambda^2 - 12\lambda = 0$$

$$\lambda(\lambda-2)(\lambda-6) = 0$$

$\therefore \lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 6$

c)

$$\lambda_1 = 0 : \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & 3 \\ 1 & 2 & 3 \end{bmatrix} \underline{u}_1 = \underline{0} \Rightarrow \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2 : \begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \underline{u}_2 = \underline{0} \Rightarrow \underline{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 6 : \begin{bmatrix} -5 & -1 & 0 \\ -1 & -2 & 3 \\ 1 & 2 & -3 \end{bmatrix} \underline{u}_3 = \underline{0} \Rightarrow \underline{u}_3 = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$$

5 a) Eigenvalues: $\lambda_1 = 1, \; \lambda_2 = -1$

Eigenvectors: $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \underline{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore \; x_h(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b) By variation of parameters:

with $x_h(t) = X(t)\,\underline{c}$, try $x_p = X(t)\,\underline{v}(t)$

$$X(t)\,\underline{v}'(t) = \begin{bmatrix} e^{t} & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \underline{v}' = \begin{bmatrix} e^{t} \\ 1 \end{bmatrix}$$

$$\therefore \; v_2' = -e^{t}$$

$$e^{t} v_1' + e^{-t} \cdot (-e^{t}) = e^{t}, \quad v_1' = 1 + e^{-t}$$

$$v_2(t) = -e^{t} \qquad v_2(t) = t - e^{-t}$$

$$x_p(t) = (te^{t} - 1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$