FINAL 2360

1. a. Separable: $\frac{y'}{y^2} = 1$, Integrate: $-\frac{1}{y} = t + C$,

$y = -\frac{1}{t+C}$. $y(0) = 1 \Rightarrow C = -1 \Rightarrow y(t) = \frac{1}{1-t}$,

This solution becomes infinite for $t = 1$,

b. Write the DE $y' - \frac{y}{t} = 1$, i.e. the integrating factor is $e^{-\int \frac{dt}{t}} = e^{-\log t} = e^{\log 1/t} = \frac{1}{t}$. Therefore

$$\frac{d}{dt}\left(\frac{1}{t} \cdot y\right) = \frac{1}{t}, \quad \frac{1}{t} y = \ln|t| + C,$$

From $y(1) = 1$ follows $C = 1$.

$\Rightarrow$ $y = t \ln|t| + t$.

Solution is valid only down to $t = 0$,

So it is $y = t \ln t + t$ for $t > 0$.

c. We recognize the DE as a Bernoulli equation. with $\alpha = 3$. Therefore, set $u = y^{1-3} = \frac{1}{y^2} \Rightarrow$

$y = u^{-1/2}$, $y' = -\frac{1}{2} u^{-3/2} u'$

Plug into the DE:

$$-\frac{1}{2} u^{-3/2} u' = \frac{u^{-1/2}}{t} + u^{-3/2}$$

which simplifies to

$$u' + \frac{2}{t} u = -2$$

Integrating factor $e^{2\int \frac{dt}{t}} = e^{2 \ln t} = e^{\ln t^2} = t^2$.

$$\Rightarrow \frac{d}{dt}(t^2 u) = -2t^2 \Rightarrow t^2 u = -\frac{2}{3} t^3 + C,$$

$$u = -\frac{2}{3} t + \frac{C}{t^2}$$

$y(1)=\sqrt{3} \Rightarrow u(1)=\frac{1}{3} \Rightarrow C=1.$

Therefore $y(t)=\frac{1}{\sqrt{u(t)}}=\frac{1}{\sqrt{\frac{1}{t^2}-\frac{2}{3}t}}=\frac{\sqrt{3}\,t}{\sqrt{3-2t^3}}.$

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2. a. According to Newton's law of cooling, the tea temperature will obey

$$\begin{cases} T'(t)=k(30-T(t)) \\ T(0)=90,\ T(1)=60. \end{cases}$$

b. We write the DE

$T'+kT=30k$; Integrating factor e^{kt}.

$\Rightarrow \frac{d}{dt}(e^{kt}\cdot T)=30ke^{kt}$

$\Rightarrow T(t)=30+C\cdot e^{-kt}.$

$T(0)=90 \Rightarrow C=60$

$T(1)=60 \Rightarrow C\cdot e^{-k}=30$; $e^{-k}=\frac{1}{2}$

The temperature at time t will therefore be $T(t)=30+60\cdot 2^{-t}.$

Eliminate using row 1.

3. a. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 \end{array}\right]$

We can immediately read off
$z = -1$, $y = 1$; then top row tells $x = 1$.

b. The elimination we just did did not change the value of the determinant. For an upper triangular matrix, it is the product of the diagonal elements. $\Rightarrow \det = 4$.

c. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 3 & -1 & 1 & -1 \end{bmatrix}$

Elimination to RREF gives

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & \tfrac{1}{2} & 2 \\ 0 & -4 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \tfrac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \tfrac{1}{2} & 1 \\ 0 & 1 & \tfrac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first <u>two</u> columns contain pivot entries. Therefore, the first two columns of A can be used for a base of the column space (dimension 2).

4. $A = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

a. $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\lambda_1 = 0$.

b. $(A - \lambda I)(\underline{v}) = \frac{1}{3}\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\left(\underline{v}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So we get
$v_1 + v_2 + v_3 = 0$ three times;
This is solved for ex by
$\left(\underline{v}_1\right) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and by $\left(\underline{v}_2\right) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

c. Part (a) showed that the matrix A is singular.
So its determinant is therefore zero.

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5. a. We guess $y = Ae^{-2t}$.
Substituting into the DE gives
$4Ae^{-2t} - Ae^{-2t} = e^{-2t}$
$3A = 1$, $A = \frac{1}{3}$
Therefore $y_p = \frac{1}{3}e^{-2t}$.

b. Homogeneous equation $y''-y=0$

$r^2-1=0 \Rightarrow r=\pm 1$.

So

$y_1(t)=e^t, \quad y_2(t)=e^{-t}$

$W(y_1,y_2)=\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t}\end{vmatrix} = -1-1=2$

$y_p = v_1(t)y_1(t)+v_2(t)y_2(t)$

where (standard formula)

$v_1' = -\dfrac{e^{-t}\,e^{-2t}}{-2} = \frac{1}{2}e^{-3t} \; ; v_1(t) = -\frac{1}{6}e^{-3t}$

$v_2' = \dfrac{e^{t}e^{-2t}}{-2} = -\frac{1}{2}e^{-t} \; ; v_2(t) = \frac{1}{2}e^{-t}$

and

$y_p = -\frac{1}{6}e^{-3t}e^{t} + \frac{1}{2}e^{-t}e^{-t} = \left(-\frac{1}{6}+\frac{1}{2}\right)e^{-2t} = \frac{1}{3}e^{-2t}$

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6 x-nullclines: $x=\pm 1, \; y=0$

y-nullclines: $y=\pm 1, \; x+\frac{3}{10}y=0$

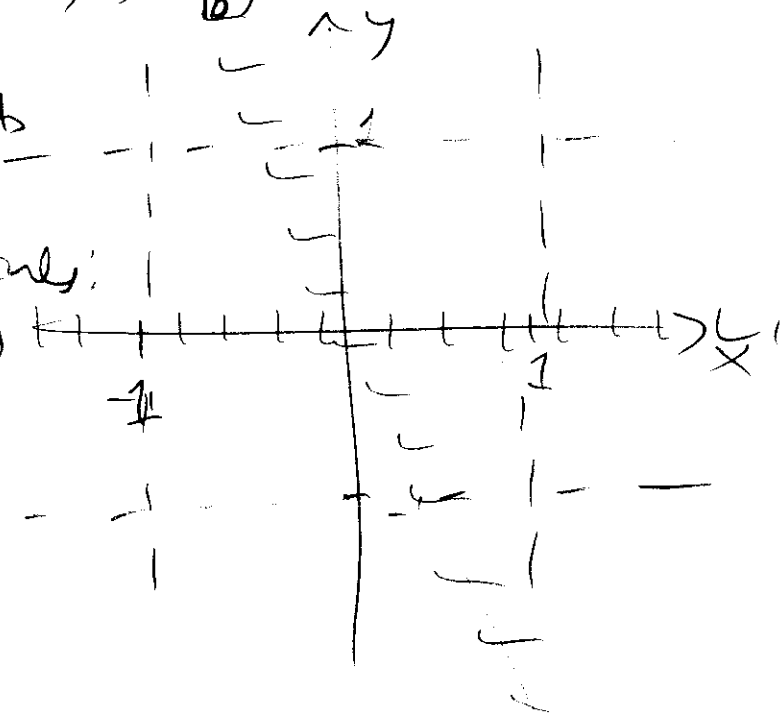
a)

So equilibrium points at all intersections of different nullclines:

$(1,1),(1,-1),(-1,1),(-1,-1)$

$(0,0)$

$(-1,\frac{10}{3}),(1,-\frac{10}{3})$.



b, Locally at origin, can ignore quadratic terms. So

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & \frac{3}{10} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Eig. values:

$$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & \frac{3}{10} - \lambda \end{vmatrix} = \lambda^2 - \frac{3}{10}\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{3}{20} \pm \sqrt{\frac{9}{400} - 1}$$

So complex, real part $> 0 \Rightarrow$ spiral out.

For direction: Check direction of arrow, for ex. $y = 0$, $x \ll 1$; Then $x' = 0$, $y' > 0$. So counterclockwise.

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7. a – 5
b – 4
c – 2
d – 3
e – 6
f – 1.

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8. Multiple choice - for each question, mark below by a cross either True or False (i.e. not always correct). Each of the 20 questions is worth 1 point, and there is a 5 point bonus if you get them all correct. You need not give any explanations.

		True	False
a.	The DE $y' = t + y$ is separable	☐	☒
b.	The DE $y' + \sqrt{t}\, y = e^{-3t}$ is linear	☒	☐
c.	The DE $y' - y = y^3$ is a Bernoulli equation	☒	☐
d.	If the rows of a matrix are linearly independent, so are the columns	☐	☒
e.	It holds that $\dfrac{a+ib}{c+id} = \dfrac{(ac+bd)-i(bc+ad)}{c^2+d^2}$	☐	☒
f.	A quadratic equation $x^2 + ax + b = 0$ with real coefficients can have a double root that is not real	☐	☒
g.	If the eigenvalues to a real matrix are complex, so are the eigenvectors	☒	☐
h.	The system $\begin{cases} x' = xy \\ y' = 1 - x^2 - y^2 \end{cases}$ has five equilibrium points	☐	☒
i.	The matrix $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ has two independent eigenvectors	☐	☒
j.	The matrix in problem 8i above is singular	☐	☒
k.	All solutions to $y'' + 4y = \cos 2t$ grow unbounded with time	☒	☐
l.	$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1$	☐	☒
m.	If A and B are square matrices, then $(AB)^T = A^T B^T$	☐	☒
n.	If $\lvert b \rvert > 1$, then solutions to $y' - by = 0$ grow with time	☐	☒
o.	The solutions to $y' = -t/y$ form circles in the (t,y)-plane	☒	☐
p.	The integrating factor for $y' + y = \dfrac{1}{1+e^t}$ is e^t	☒	☐
q.	$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$	☐	☒
r.	If A is invertible (non-singular), the linear system $Ax = b$ has a unique solution	☒	☐
s.	Vectors that form a base are always linearly independent	☒	☐
t.	$\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector to the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	☒	☐