APPM 2360–EXAM I **7.00-8.30pm February 15th 2006**

On the front of your blue book, write your name, the names of your lecturer (or lecture session number) and of your TA (or recitation section number). Draw also a grading grid. There are FIVE problems (with subparts a, b, ...). You must solve all five problems. Each full problem is worth 20 points. Start each problem on a new page. Show all your work in your bluebook. Explain all steps in your solutions. Box all your answers. Calculators, books or any notes are NOT permitted, with the exception of one two-sided $8.5'' \times 11''$ 'crib sheet'

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1. Consider the following ordinary differential equation (ODE)

$$y' + \frac{2}{t}y = 4t$$

Using the method of integrating factors

   (a) Find an integrating factor $\mu(t)$.

   (b) Find the general solution to the ODE.

   (c) Find the solution with initial condition $y(1) = 2$.

Answer:

Linear ODE of form $y' + p(t)y = f(t)$, thus $p(t) = 2/t$ and $f(t) = 4t$

a) Integrating factor:

$$\mu(t) = e^{\int p(t)dt} = e^{\int 2/tdt} = t^2.$$

b) Multiply ODE by $\mu(t) = t^2$ to get

$$\frac{d}{dt} = \big(t^2 y(t)\big) = 4t^3.$$

Integrate w.r.t. t for general solution

$$y(t) = t^2 + \frac{C}{t^2}.$$

c) Apply initial condition $y(1) = 2$

$$2 = 1 + C \quad \Rightarrow \quad C = 1.$$

Particular solution is given by

$$y(t) = t^2 + \frac{1}{t^2}.$$

2. Consider the initial value problem (IVP)

$$y' = [(t-1)y]^{\frac{1}{3}}, \quad y(0) = y_0,$$

where y_0 is an unspecified constant.

(a) Does Picard's Theorem apply to the given IVP with $y_0 = 0$? Show your work in determining your answer.

(b) Does Picard's Theorem apply to the given IVP with $y_0 = 1$? Show your work in determining your answer.

(c) If your answer to (b) is YES, solve the IVP.

Answer:

Picards Theorem requires

$$f(t,y) = [(t-1)y]^{\frac{1}{3}} \tag{1}$$

and

$$\partial_y f(t,y) = \frac{1}{3}(t-1)^{\frac{1}{3}} y^{-\frac{2}{3}} \tag{2}$$

be well-defined at initial condition (t_0, y_0)

a) for initial condition $(0,0)$ (1) is satisfied (2) is not. Picard's Thm does not hold.

b) for initial condition $(0,1)$ (1) and (2) are satisfied. Picard's Thm holds.

c) ODE is nonlinear and separable

$$\begin{aligned} \int y^{-\frac{1}{3}} dy &= \int (t-1)^{\frac{1}{3}} dt, \\ \Rightarrow \frac{3}{2} y^{\frac{2}{3}} &= \frac{3}{4}(t-1)^{\frac{4}{3}} + C. \end{aligned}$$

Impose initial condition $(0,1)$ to get $C = \frac{3}{4}$ and

$$y(t) = \left[\frac{1}{2}\left((t-1)^{\frac{4}{3}} + 1\right)\right]^{\frac{3}{2}}.$$

3. Consider the following initial value problem (IVP):

$$y' = \frac{2y}{1+t}, \qquad y(0) = 1$$

Using Euler's method

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + hf(t_n, y_n) \end{aligned}$$

Find the approximation to $y(1)$ using $h = 1/2$.

n	t_n	$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$
0	$t_0 = 0$	$y_0 = 1$
1	$t_1 = \frac{1}{2}$	$y_1 = 1 + \frac{1}{2}\left(\frac{2\cdot 1}{1+0}\right) = 2$
2	$t_2 = 1$	$y_2 = 2 + \frac{1}{2}\left(\frac{2\cdot 2}{1+\frac{1}{2}}\right) = 3\frac{1}{3}$

4. Initially a tank contains 100 liters of fresh water. Fluid enters the tank from two pipes. One pipe pumps in pure water at 1/2 a liter per minute while the other pumps in salt water with a concentration of 2 kilograms of salt per liter at the rate of 1/2 a liter per minute. A well mixed solution leaves the tank at the rate of 2 liters per minute.

   (a) Derive the IVP for $y(t)$, the number of kilograms of salt at time t.

   (b) Solve the initial value problem.

   (c) Does the amount of salt in the tank ever reach a maximum value? If so, derive an expression for the maximum amount of salt in the tank.

Answer:

a) The concentration of salt entering the tank is 2 kg/l and its flow rate is $\frac{1}{2}$ l/min. The net outflow of the fluid is 2 l/min - $(\frac{1}{2}+\frac{1}{2})$ l/min = 1 l/min. The volume of fluid in the tank is changing with time as $\frac{1}{100-t}$. Therefore, the concentration of salt flowing out of the tank is $\frac{y(t)}{100-t}$. The equation of continuity gives

$$\begin{aligned} y' &= \text{Rate In} - \text{Rate Out} \\ &= (\text{Concentration In} \times \text{Flow Rate In}) - (\text{Concentration Out} \times \text{Flow Rate Out}) \\ &= (2 \text{ kg/l}) \times (\frac{1}{2} \text{ l/min}) - (\frac{y}{100-t} \text{ kg/l}) \times (2 \text{ l/min}) \\ &= 1 - \frac{2y}{100-t}. \end{aligned}$$

b) The *general* solution by variation of parameters or integrating factors is

$$y(t) = 100 - t + c(100-t)^2.$$

The initial condition is $y(0) = 0$ so the solution to the IVP takes the form

$$y(t) = 100 - t - \frac{(100-t)^2}{100}.$$

c) Yes, the maximum value occurs when $y'(t_{max}) = 0$ or

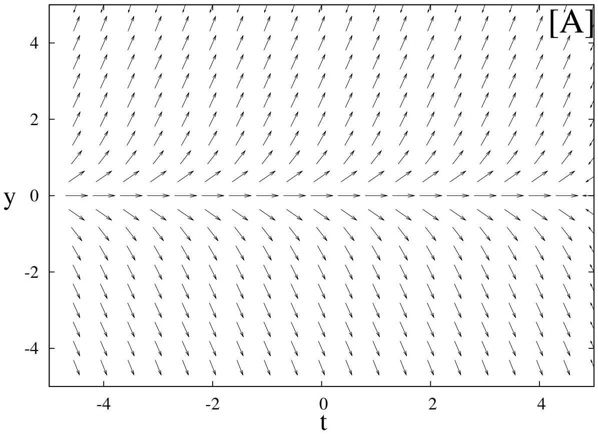
$$t_{max} = 50.$$

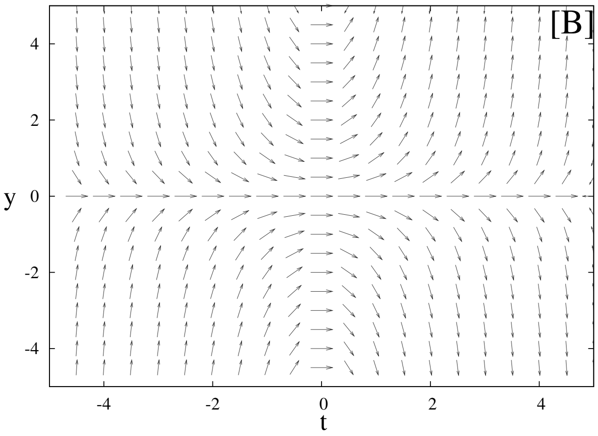
with the value

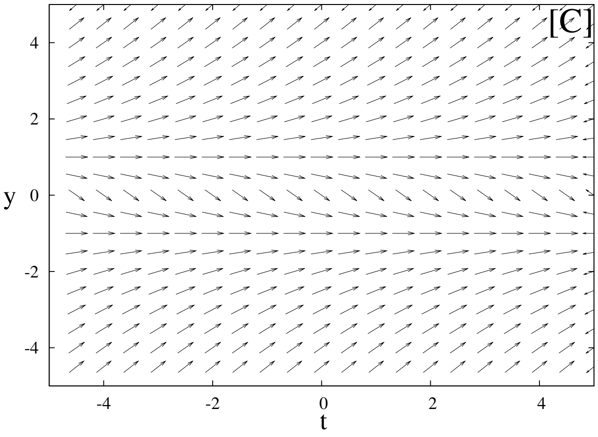
$$y_{max} = y(t_{max}) = 100 - t_{max} - \frac{(100-t_{max})^2}{100} = 25.$$

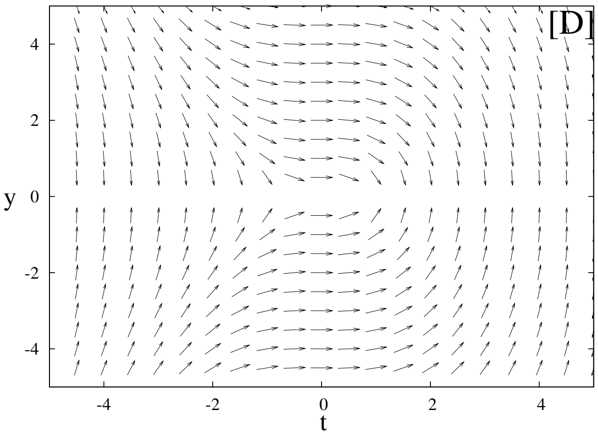
5. Match the following differential equations [1]–[4] with their corresponding direction fields [A]–[D].

$$[1]\ \frac{dy}{dt} = y + \sin(y), \quad [2]\ \frac{dy}{dt} = \sqrt[+]{|y|} - 1, \quad [3]\ \frac{dy}{dt} = \frac{-t^2}{y}, \quad [4]\ \frac{dy}{dt} = ty$$









Answer:

[A] ↔ 1

[B] ↔ 4

[C] ↔ 2

[D] ↔ 3