APPM 2360–EXAM II **7.00-8.30pm March 15th 2006**

On the front of your blue book, write your name, the names of your lecturer (or lecture session number) and of your TA (or recitation section number). Draw also a grading grid. There are FIVE problems (with subparts a, b, ...). You must solve all five problems. Each full problem is worth 20 points. Start each problem on a new page. Show all your work in your bluebook. Explain all steps in your solutions. Box all your answers. Calculators, books or any notes are NOT permitted, with the exception of one two-sided $8.5'' \times 11''$ 'crib sheet'

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1. a. Convert the linear system

$$\begin{aligned} z + 3x &= 1 \\ 2y - x + 2z &= -1 \\ -z + 5x - 3y &= 1 \end{aligned}$$

into matrix-vector form $\mathbf{Ax} = \mathbf{b}$. Identify $\mathbf{A}, \mathbf{b}$.

b. From your answer in a. calculate $\mathbf{b}^T\mathbf{A}^T$ and verify that it is identical to $(\mathbf{Ab})^T$.

c. Which of the following matrix mutiplication exists: (i) $\mathbf{A}^T\mathbf{b}$ (ii) $\mathbf{A}\mathbf{b}^T$ (iii) $\mathbf{b}^T\mathbf{A}$

Answer:

a)

$$\begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 2 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

b) $\mathbf{b}^T\mathbf{A}^T = [4 \ -1 \ 7]$

c) (i), (iii)

2. a. If possible, use Gaussian elimination to solve the following system of linear equations for x, y, and z:

$$\begin{aligned} 3x + z &= 1 \\ -x + 2y + 2z &= -1 \\ 5x - 3y - z &= 1 \end{aligned}$$

If it is not possible, give a reason.

b. Solve the matrix, vector equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

Answer:

a) $(x, y, z) = (1, 2, -2)$.

b) $(x_1, x_2) = (\frac{1}{2} - \frac{1}{2}t, t)$ for any real number t.

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 16 & 0 & 4 & 7 \\ -3 & 8 & 8 & 2 \\ 1 & 0 & 5 & 2 \\ -7 & 6 & 5 & 4 \end{bmatrix}$$

a. Does the matrix **A** have an inverse ?

b. Write down the RREF of **A** without performing any computations.

c. Characterize the solutions of $\mathbf{Ax} = \mathbf{0}$.

d. Characterize the solutions of $\mathbf{Ax} = \mathbf{b}$.

Answer:

a).

$$|A| = 8C_{22} + 6C_{42}$$

$$C_{22} = (-1)^4 \begin{vmatrix} 16 & 4 & 7 \\ 1 & 5 & 2 \\ -7 & 5 & 4 \end{vmatrix} = 368, \quad C_{42} = (-1)^6 \begin{vmatrix} 16 & 4 & 7 \\ -3 & 8 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -33,$$

Therefore, $|A| = 8(368) - 6(33) = 2746 \neq 0$ and $\mathbf{A}^{-1}$ exists.

b)

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) Because **A** is non-singular and square, the solution consists of the zero vector $\mathbf{x} = 0$.

d) Because **A** is non-singular and square, the unique solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

4. Given the vectors:

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

a. Are these vectors linearly independent ?

b. Define the two properties of a basis. Do these vectors form a basis ?

c. Find three coefficients a_i such that

$$\mathbf{z} = \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} = a_1\mathbf{y}_1 + a_2\mathbf{y}_2 + a_3\mathbf{y}_3.$$

Answer:

a)

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4 - 1 + 0 - 1 - 0 = 2$$

Therefore the vectors are linearly independent.

b) A basis is linearly independent and spans the vector space. These are three linearly independent vectors that span $\mathbf{R}^3$ and thus form a basis.

c)

$$\mathbf{z} = \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} = a_1\mathbf{y}_1 + a_2\mathbf{y}_2 + a_3\mathbf{y}_3$$

Form the system of equations

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix}$$

Row reduce to

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 5 \end{bmatrix}$$

and back-solve or find the RREF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -13/2 \\ 5 \end{bmatrix}$$

so the coefficients are

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -13/2 \\ 5 \end{bmatrix}$$

5. Consider the second-order ordinary differential equation (ODE):

$$y'' - 2y' + y = 0$$

a. Find two linearly independent solutions $y_1(t)$ and $y_2(t)$.

b. Calculate the Wronskian of $y_1(t)$ and $y_2(t)$. What is the dimension of the vector space spanned by $y_1(t)$ and $y_2(t)$?

c. Using $y_1(t)$ and $y_2(t)$ construct the general solution $y(t)$ of the ODE.

d. Find the solution with the conditions $y(0) = 1$ and $y(1) = 0$.

Answer:

a) The ODE is linear and has constant coefficients, we can solve the equation by finding the characteristic equation, putting $y = \exp(rt)$

$$r^2 - 2r + 1 = 0 \quad (1)$$

$$(r-1)^2 = 0 \quad (2)$$

There is therefore a repeated root, $r_1 = r_2 = 1$, and the first solution is $y_1(t) = \exp(t)$. To find the second linearly independent solution we look for solutions of the form $y_2(t) = v(t)\exp(t)$ for some function $v(t)$. Substituting into the ODE we get $v''(t) = 0$ and our second solutions is $y_2(t) = t\exp(t)$.

b) The Wronskian of $y_1(t)$ and $y_2(t)$ is:

$$\mathrm{Wr}[e^t, te^t] = \begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} \quad (3)$$

$$= (1+t)e^{2t} - te^{2t} = e^{2t} \quad (4)$$

As $\mathrm{Wr}[e^t, te^t] \neq 0$ the vectors y_1 and y_2 are linearly independent and are basis vectors for the solution space. This vector space is therefore two dimensional.

c) The general solution is the span of the individual solutions.

$$y(t) = c_1 y_1 + c_2 y_2 = c_1 e^t + c_2 t e^t$$

d) With the boundary conditions $y(0) = 1$ and $y(1) = 0$.

$$y(0) = c_1 \times e^0 + c_2 \times 0 \times e^0 = c_1 = 1$$

Therefore $c_1 = 1$

$$y(1) = c_1 e + c_2 e = 0$$

Therefore $c_2 = -c_1 = -1$ and the solution is:

$$y(t) = e^t - te^t$$