APPM 2360–EXAM III **7.00-8.30pm April 19th 2006**

On the front of your blue book, write your name, the names of your lecturer (or lecture session number) and of your TA (or recitation section number). Draw also a grading grid. There are FIVE problems (with subparts a, b, ...). You must solve all five problems. Each full problem is worth 20 points. Start each problem on a new page. Show all your work in your bluebook. Explain all steps in your solutions. Box all your answers. Calculators, books or any notes are NOT permitted, with the exception of one two-sided $8.5'' \times 11''$ 'crib sheet'

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1. For each of the following equations write down the form of the particular solution according to the method of undetermined coefficients **if it is possible to use this method** (you do NOT need to find the values of the coefficients). If it is not possible to use the method of undetermined coefficients, state another method that could be used to solve for the particular solution (you do NOT need to find the particular solution).

   a) $y'' + 3y' + 2y = t^2$

   b) $y'' + 3y' + 2y = \frac{1}{3}e^{-t}$

   c) $y'' + 3y' + 2y = te^t$

   d) $y'' + y = \tan(t)$

   Solution:

   a) $y'' + 3y' + 2y = t^2$:

   $$y_p(t) = At^2 + Bt + C,$$

   where A, B, and C are constants to be determined.

   b) $y'' + 3y' + 2y = \frac{1}{3}e^{-t}$:
   Since one of the homogenous solutions is e^{-t}, we seek a particular solution of the form

   $$y_p(t) = Ate^{-t},$$

   where A is a constant to be determined.

   c) $y'' + 3y' + 2y = te^t$:

   $$y_p(t) = (At + B)e^t.$$

   d) $y'' + y = \tan(t)$:
   $\tan(t)$ is NOT in the class of elementary functions solvable by undetermined coefficients. A particular solution can be found using variation of parameters.

2. a) Find the general solution to the following non-homogeneous ordinary differential equation (ODE), using variation of parameters to find the particular solution.

$$y''(t) + 5y'(t) + 4y(t) = \sinh(t)$$

Where $\sinh(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$.

b) For the initial conditions $y(0) = \frac{1}{5}$, $y'(0) = \frac{1}{30}$ find the specific solution to the ODE in part (a).

Solutions:

a) To solve the associated homogeneous equation:

$$y_h'' + 5y_h' + 4y_h = 0$$

Assuming a solution of the form $y_h = e^{rt}$ we get the characteristic equation, $r^2 + 5r + 4 = 0$ with roots $r_1 = -1$ and $r_2 = -4$

The homogeneous solution is therefore:

$$y_h = c_1 e^{-t} + c_2 e^{-4t}$$

Using variation of parameters to find the particular solution, look for solutions of form

$$y_p = v_1(t)e^{-t} + v_2(t)e^{-4t}$$

Which gives the two equations:

$$v_1' e^{-t} + v_2' e^{-4t} = 0$$
$$-v_1' e^{-t} - 4v_2' e^{-4t} = \sinh(t)$$

Solving the simultaneous equations for v_1 and v_2:

$$v_1(t) = \frac{1}{3}\int e^t \sinh(t)dt = \frac{1}{12}e^{2t} - \frac{t}{6}$$
$$v_2(t) = -\frac{1}{3}\int e^{4t} \sinh(t)dt = -\frac{1}{30}e^{5t} + \frac{1}{18}e^{3t}$$

The general solution is thus given by:

$$y(t) = y_h + y_p = c_1 e^{-t} + c_2 e^{-4t} + \frac{1}{20}e^t - \frac{t}{6}e^{-t}$$

b) With the initial conditions $y(0) = \frac{1}{5}$, $y'(0) = \frac{1}{30}$ the general ODE gives the relations:

$$y(0) = c_1 + c_2 + \frac{1}{20} = \frac{4}{20}$$
$$y'(0) = -c_1 - 4c_2 + \frac{1}{20} - \frac{1}{6} = \frac{2}{60}$$

When solved simultaneously this gives the values for the constants $c_1 = \frac{1}{4}$ and $c_2 = -\frac{1}{10}$.

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

a) Find the eigenvalues and eigenvectors of this matrix.

b) What is the dimension of the eigenspace for each eigenvalue ?

Solution:

a) The matrix is upper-triangular. Therefore, it has has two distinct eigenvalues $\lambda_{1,2} = 1$ and $\lambda_3 = 2$.

or

The characteristic polynomial of $\mathbf{A}$ is given by $|\mathbf{A} - \lambda\mathbf{I}| = p(\lambda) = 0$. In this case

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = p(\lambda) = (1-\lambda)^2(2-\lambda) = 0$$

Therefore, the matrix $\mathbf{A}$ has two distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

An eigenvector $\mathbf{v}_1$ is a solution of

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies $v_3 = 0$, and $v_1 = s$ and $v_2 = t$ are free parameters. Therefore, the general solution is given by

$$\mathbf{v} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We therefore find two eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

An eigenvector $\mathbf{v}_3$ is a solution of

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $v_3 = s$, then $v_2 = 4s$, $v_1 = 3s$ and an eigenvector is

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

b) E_{λ_1} is spanned by two eigenvectors therefore $\dim(E_{\lambda_1}) = 2$. E_{λ_3} is spanned by one vector, $\dim(E_{\lambda_2}) = 1$.

4. Consider the linear system of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find the homogeneous solution $\mathbf{x}(t)$.

b) Sketch the phase plane diagram for this problem.

c) The solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Find the solution $\mathbf{x}(t)$, and the matrix exponential $e^{\mathbf{A}t}$, given the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Note $e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$ where $\mathbf{X}(t)$ is the fundamental matrix of solutions at time t.

Solution:

a) $\lambda_1 = -4$ and $\lambda_2 = 2$ are the eigenvalues of $\mathbf{A}$, and these are the roots of

$$\mathbf{A} - \lambda\mathbf{I} = \begin{vmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 = (\lambda+4)(\lambda-2) = 0$$

The first eigenvector $\mathbf{v}_1$ is the solution of

$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying Gaussian elimination

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the eigenvector is $\mathbf{v}_1 = [1, -4]^T$.

The second eigenvector is a solution of

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After Gaussian elimination

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the eigenvector is $\mathbf{v}_2 = [1, 2]^T$.

Therefore, the homogeneous solution is a linear combination of the basis vectors

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

given by

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This can be expressed as the matrix-vector product

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where $\mathbf{X}(t)$ is the fundamental matrix

$$\mathbf{X}(t) = \begin{bmatrix} e^{-4t} & e^{2t} \\ -4e^{-4t} & 2e^{2t} \end{bmatrix}$$

c) The initial condition satisfies the equation $\mathbf{x}(0) = \mathbf{x}_0 = \mathbf{X}(0)\mathbf{c}$, given by

$$\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The inverse of $\mathbf{X}(0)$ is given by

$$\mathbf{X}^{-1}(0) = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}$$

and therefore

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The solution for this initial condition is then given by

$$\mathbf{x}(t) = (1) \cdot e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + (0) \cdot e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ -4e^{-4t} \end{bmatrix}$$

The solution can also be expressed as $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0)\mathbf{x}_0$

$$\mathbf{x}(t) = \begin{bmatrix} e^{-4t} & e^{2t} \\ -4e^{-4t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ -4e^{-4t} \end{bmatrix}$$

The solution of the homogeneous problem $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

where the matrix exponential is given by

$$e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$$

For the given initial value problem

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} e^{-4t} & e^{2t} \\ -4e^{-4t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2e^{-4t} + 4e^{2t} & -e^{-4t} + e^{2t} \\ -8e^{-4t} + 8e^{2t} & -4e^{-4t} + 2e^{2t} \end{bmatrix}$$

5. Consider the following initial value problem:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} \frac{4}{3} & -\frac{5}{3} \\ \frac{5}{3} & -\frac{4}{3} \end{pmatrix}$$

a) Find the eigenvalues of the matrix **A**.

b) Find the eigenvectors of the matrix **A**.

c) Construct two real-valued, linear independent solutions to the system.

d) From your answer to c) solve the initial value problem.

Solution:

a) $\lambda_1 = +i$ and $\lambda_2 = \overline{\lambda}_1$ are the eigenvalues of $\mathbf{A}$, and these are the roots of

$$\mathbf{A} - \lambda\mathbf{I} = \lambda^2 + 1 = 0$$

b) The first eigenvector $\mathbf{v}_1$ is the solution of

$$\begin{pmatrix} \frac{4}{3} - i & -\frac{5}{3} \\ \frac{5}{3} & -\frac{4}{3} - i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives an eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 4 - 3i \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$\mathbf{v}_2 = \overline{\mathbf{v}}_1$

c) The complex-valued $\mathbf{z}_1 = \mathbf{u}(t) + i\mathbf{v}(t)$ is given by

$$\begin{aligned} \mathbf{z}_1 &= e^{it} \left[\begin{pmatrix} 5 \\ 4 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right] \\ &= \left[\cos t \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right] + i \left[\cos t \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \sin t \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right] \end{aligned}$$

The real-valued, linearly independent solutions are $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

d) General solution: $\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$. For the given initial value problem, solve

$$\begin{bmatrix} 5 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{15} \end{bmatrix}$$