1 (a) $\frac{dw}{dx} = w^2 \sin(x) \qquad w(0) = 1/5$

$\Rightarrow \cancel{\int \frac{dw}{w^2} = \int \sin(x)\,dx}$

$\Rightarrow \frac{-1}{w} = -\cos(x) + C \quad \Rightarrow \frac{1}{w} = \cos(x) + C \qquad (c \to -c)$

$\Rightarrow w = \frac{1}{C + \cos(x)}$

$\Rightarrow w(0) = \frac{1}{C+1} = \frac{1}{5} \quad \Rightarrow C = 4$

$\Rightarrow \boxed{w(x) = \frac{1}{4+\cos(x)}}$

(b) $x' - \tan(t)x = t \qquad$ Int. Factor: $\mu = e^{\int -\tan t \, dt} = e^{-\int \frac{\sin t}{\cos t} dt} = e^{\log(\cos t)}$

$= \cos t$

$\Rightarrow x'\cos(t) - \sin(t)x = t\cos(t)$

$\Rightarrow [x\cos(t)]' = t\cos(t)$

$\Rightarrow x\cos(t) = \int t\cos(t)\,dt = t\sin(t) - \int \sin(t)\,dt$

$= t\sin(t) + \cos(t) + C$

$\Rightarrow x(t) = t\tan(t) + 1 + \frac{C}{\cos(t)}$

$x(0) = 0 + 1 + \frac{C}{1} = 0 \quad \Rightarrow C = -1$

$\Rightarrow \boxed{x(t) = t\tan(t) + 1 - \sec(t)}$

2. Measure t in hours with $t=0$ corresponding to 11:30 am

$$T' = k(T_r - T) \Rightarrow T' + kT = kT_r$$

$$\Rightarrow T'e^{kt} + Tke^{kt} = kT_r e^{kt} \quad \text{using } \mu = e^{\int k\,dt} = e^{kt}$$

$$\Rightarrow (Te^{kt})' = T_r ke^{kt}$$

$$\Rightarrow Te^{kt} = T_r \int ke^{kt}\,dt = T_r e^{kt} + C$$

$$\Rightarrow T = T_r + Ce^{-kt}$$

$$= 70 + Ce^{-kt}$$

$$T(0) = 70 + C = 94.6 \Rightarrow C = 24.6$$

$$\Rightarrow \boxed{T(t) = 70 + 24.6e^{-kt}}$$

$$T(1) = 70 + 24.6e^{-k} = 93.4 \Rightarrow e^{-k} = \frac{23.4}{24.6} \Rightarrow \boxed{k = \log\left(\frac{24.6}{23.4}\right)}$$

Time of death is t such that $T(t) = 70 + 24.6e^{-kt} = 98.6$

$$\Rightarrow e^{-kt} = \frac{28.6}{24.6} \Rightarrow t = \frac{1}{k}\log\left(\frac{24.6}{28.6}\right)$$

$$\Rightarrow \boxed{t = \frac{\log\left(\frac{24.6}{28.6}\right)}{\log\left(\frac{24.6}{23.4}\right)} = \frac{\log(24.6) - \log(28.6)}{\log(24.6) - \log(23.4)}}$$

This is a negative number (time of death is prior to 11:30 am)

$\Rightarrow$ $\boxed{\text{time of death is } \left|\frac{\log\left(\frac{24.6}{28.6}\right)}{\log\left(\frac{24.6}{23.4}\right)}\right| \text{ hours before 11:30 am.}}$

3 (a) $y(t) = 0$ is a solution because $y' = 0$ & $\sin\sqrt{y}\, e^{-t^2} = \sin(\sqrt{0})e^{-t^2} = 0$

& clearly $y(0) = 0$.

(b) $\left.\begin{aligned} \mathcal{L}[u] &= f(t) \\ \mathcal{L}[v] &= g(t) \\ \mathcal{L}[w] &= 0 \end{aligned}\right\} \Rightarrow \mathcal{L}[u] + \pi \mathcal{L}[v] + C\mathcal{L}[w] = f(t) + \pi g(t) + C\cdot 0 = h(t)$

(for any constant C)

By linearity $\mathcal{L}[u] + \pi\mathcal{L}[v] + C\mathcal{L}[w] = \mathcal{L}[u + \pi v + Cw] = h(t)$

so $\boxed{y = u + \pi v + Cw}$ is general sol$\underline{\text{n}}$ to $\mathcal{L}[y] = h(t)$

(c) $x' = |x| - t^2 = f(t,x)$

$f(t,x)$ is continuous for all (t,x) so, by Picard's Theorem $x' = f(t,x)$ $\boxed{\text{has solutions}}$ (at least one through every initial condition).

But $\frac{\partial f}{\partial x}$ is discontinuous at $x=0$, so $\boxed{\text{solutions could cross}}$ at $x = 0$ (but nowhere else), but Picard's Th$\underline{\text{m}}$ does not apply so we $\boxed{\text{don't know for sure}}$ whether they do or not.

(d) $y' = y\cos(\pi y/4)$ has equilibria at $y = 0$ & $y = \pm 2, \pm 6, \pm 10, \ldots$

Since $y(0) = 1$, the solution is bounded between $y=0$ & $y = 2$ for all t (technically because solutions can't cross, by Picard's Th$\underline{\text{m}}$)

When $y = 1$, $y' = \cos(\pi/4) > 0$ $\Rightarrow$ solutions are increasing between 0 & 2 $\Rightarrow$ $\boxed{y(t) \to 2}$ as $t \to \infty$

4 (a) $z = At + B \Rightarrow z' = A \Rightarrow z'' = 0$

So $z'' = 0 = \frac{2t}{t^2+4}(z'-1) = \frac{2t}{t^2+4}(A-1) \Rightarrow A - 1 = 0 \Rightarrow A = 1$

B can be anything, so take $B = 0 \Rightarrow \boxed{z_p(t) = t}$

(b) $z(t) = 1 \Rightarrow z' = z'' = 0$

Homogeneous eq$^{\underline{n}}$ is $z'' = \left(\frac{2t}{t^2+4}\right)z'$

$\Rightarrow 0 = \left(\frac{2t}{t^2+4}\right)0 \Leftrightarrow 0 = 0$ ✓

So $z(t) = 1$ is a solution.

(c) $y^* = z' \Rightarrow y' = z''$ so $z'' = \left(\frac{2t}{t^2+4}\right)z' \Leftrightarrow y' = \left(\frac{2t}{t^2+4}\right)y$

$\Rightarrow y = Ce^{\int \frac{2t}{t^2+4}\,dt} = Ce^{\log(t^2+4)} = C(t^2+4)$

$\Rightarrow z^* = \int y\,dt = \int C(t^2+4)\,dt$

$\Rightarrow \boxed{z = C\left(\frac{t^3}{3} + 4t\right) + D}$

(d) By linearity, general solution is $C\left(\frac{t^3}{3} + 4t\right) + \underbrace{D + A.1}_{} + z_p(t)$

$= C\left(\frac{t^3}{3} + 4t\right) + B + t$

$\Rightarrow \boxed{z(t) = C_1\left(\frac{t^3}{3} + 4t\right) + C_2 + t}$

5 (a) 6

(b) 7

(c) 3

(d) 9