1 a) FALSE
b) FALSE
c) TRUE
d) TRUE
e) FALSE

2 (a) $\lambda x + z = 3$
$2x - 3y = 4$
$x - y + z = 2$

$$\iff \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ \lambda & 0 & 1 & 3 \\ 2 & -3 & 0 & 4 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - \lambda R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & \lambda & 1-\lambda & 3-2\lambda \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{R_3 \to -R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & \lambda & 1-\lambda & 3-2\lambda \end{array}\right] \xrightarrow{R_3 - \lambda R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1-3\lambda & 3-2\lambda \end{array}\right]$$

$\Rightarrow$ If $\boxed{\lambda \neq \frac{1}{3}}$ then there is a unique sol^{n} $\left(z = \frac{3-2\lambda}{1-3\lambda} \Rightarrow y = -2z = \text{etc} \right)$

ie unique sol^{n} for all λ except $\lambda = 1/3$

(b) If $\lambda = 1/3$ then reduced system is $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 7/3 \end{array}\right]$

which is inconsistent $\Rightarrow$ no sol^{n}
ie no sol^{n} when $\boxed{\lambda = 1/3}$

(c) Above two cases cover every value of λ $\Rightarrow$ $\boxed{\text{no value of } \lambda}$ gives infinitely many sol^{n}s.

3 (a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ unless $ad = bc$

$\Rightarrow A^{-1} = \frac{1}{5-6}\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}}$

(b) $A\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \vec{x} = A^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ -4 \end{bmatrix}}$

(c) $\boxed{\text{No}}$ Given any vector $\vec{b}$, $\vec{x} = A^{-1}\vec{b}$ is the unique sol$^{\underline{n}}$ b/c A^{-1} exists (& is unique)

---

4 $x'' - 2x' - 3x = 0$

(a) $x = e^{rt} \Rightarrow x'' - 2x' - 3x = e^{rt}(r^2 - 2r - 3) = 0$

$\Rightarrow (r-3)(r+1) = 0 \Rightarrow r = 3, -1$

$\Rightarrow$ general sol$^{\underline{n}}$ is $x(t) = c_1 e^{3t} + c_2 e^{-t}$

(b) $x(0) = c_1 + c_2 = \alpha$

$x' = 3c_1 e^{3t} - c_2 e^{-t} \Rightarrow x'(0) = 3c_1 - c_2 = \beta$ $\Bigg\} \Rightarrow$ $\begin{matrix} c_1 + c_2 = \alpha \\ 3c_1 - c_2 = \beta \end{matrix}$

or $\boxed{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}}$

(c) $\boxed{\text{No}}$ – for any α, β we can solve the system in (b):

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ The inverse exists because $\det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = -1-3 = -4 \neq 0$

5 (a) Find sol$^{\underline{ns}}$ of $c_1\begin{bmatrix}\alpha\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\\alpha\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\\alpha\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Leftrightarrow \begin{bmatrix}\alpha&1&0\\1&\alpha&1\\0&1&\alpha\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

If $\det\begin{bmatrix}\alpha&1&0\\1&\alpha&1\\0&1&\alpha\end{bmatrix}\neq 0$ then unique sol$^{\underline{n}}$ is $\begin{bmatrix}c_1\\c_2\\c_2\end{bmatrix}=\begin{bmatrix}\;\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow$ vectors are independent.

If $\det(\;)=0$ then there are infinitely many sol$^{\underline{ns}}$ (can't be unique sol$^{\underline{n}}$ & can't be no sol$^{\underline{n}}$ since $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ is always a sol$^{\underline{n}}$)

$\Rightarrow$ there is a linear combination (other than $c_1=c_2=c_3=0$) that gives the zero vector $\Rightarrow$ vectors are dependent

$$\det\begin{bmatrix}\alpha&1&0\\1&\alpha&1\\0&1&\alpha\end{bmatrix}=\alpha\begin{vmatrix}\alpha&1\\1&\alpha\end{vmatrix}-1\begin{vmatrix}1&1\\0&\alpha\end{vmatrix}=\alpha(\alpha^2-1)-\alpha$$
$$=\alpha(\alpha^2-2)$$

$\Rightarrow \det(\;)=0 \Leftrightarrow \boxed{\alpha=0 \text{ OR } \alpha=\pm\sqrt{2}}$

(b) $x+y+z=0$
$y-z=0$
$\Leftrightarrow \left[\begin{array}{ccc|c}1&1&1&0\\0&1&-1&0\end{array}\right]$ is already row reduced

$\Rightarrow z=\text{anything}=t$ Then $y=z=t$ & $x=-y-z=-t-t=-2t$

$\Rightarrow \boxed{\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2t\\t\\t\end{bmatrix}=t\begin{bmatrix}-2\\1\\1\end{bmatrix}}$ for any $t\in\mathbb{R}$.

(c) The set of all sol$^{\underline{ns}}$ is $t\begin{bmatrix}-2\\1\\1\end{bmatrix}=\text{Span}\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}=W$

So $\boxed{\begin{bmatrix}-2\\1\\1\end{bmatrix} \text{ is a basis of } W}$ (one vector must be independent & it spans W)

$\Rightarrow \boxed{W \text{ is one-dimensional}}$ (one basis vector)

6. Let B & C both be inverses of A (different, if possible)

Then, by definition of inverse, $AB = BA = I$ & $AC = CA = I$

$\Rightarrow AB = I = AC$

$\Rightarrow B(AB) = B(AC)$

$\Rightarrow (BA)B = (BA)C$ (Associativity of matrix multiplication)

$\Rightarrow IB = IC$ $(BA = I)$

$\Rightarrow B = C$

So "both" inverses are, in fact, equal. $\Rightarrow$ the inverse is unique

Q.E.D.