INSTRUCTIONS: Computers, calculators, books, notes, fleeing monkeys, *etc.* are not permitted. Write your name, your instructor's name, and your recitation section number on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. True or False. State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You MUST write the full word TRUE or FALSE — T/F will NOT be graded. For this question only you do NOT need to show your working or reasoning.
   (a) If A does not have an inverse, then $\lambda = 0$ is an eigenvalue of A.
   (b) If $\det(A) > 0$ then the origin must be an unstable equilibrium of $\mathbf{x}' = A\mathbf{x}$.
   (c) If A is invertible and λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.
   (d) An n^{th}-order ODE in normal form (*i.e.* $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$) can always be rewritten as an equivalent n-dimensional first-order system of ODEs.
   (e) If A is $n \times n$ and $\det(A) \geq 1$, then A must have n linearly independent eigenvectors.

2. Consider the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find the general solution of $\mathbf{x}' = A\mathbf{x}$.
   (c) Show that your answer to (b) is indeed a solution by substituting it back into the differential equation.
   (d) What is the long-term behavior of the solution (for general initial conditions)? Specifically, does $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$, or does $|\mathbf{x}(t)| \to 0$, or something else?

3. Consider the differential equation $y'' - 8y' + 15y = \phi(t)$.
   (a) Find the general solution of the homogeneous part.
   (b) For each of the following $\phi(t)$, write down the form of a particular solution to the nonhomogeneous differential equation. Do not solve for any constants, just write down the form of y_p (for example "$y_p(t) = \alpha t + \beta$, where α and β are constants").
      i. $\phi(t) = 13e^{5t}$
      ii. $\phi(t) = t\cos(3t)$
      iii. $\phi(t) = -3t^2$
      iv. $\phi(t) = \frac{1}{2}\cos(3t) - \pi e^{-3t}$
      v. $\phi(t) = 2t + 3e^{8t}$

4. Consider the equation $x'' + 2x' + 10x = 3e^{-t}$.
   (a) Find the general solution of the homogeneous part.
   (b) Find the general solution of the full equation.
   (c) Find the particular solution (of the full equation) that satisfies the initial condition $x(0) = 1/3$, $x'(0) = 0$.
   (d) Sketch the solution from (c). What is the long-term ($t \to \infty$) behavior? Are there any solutions with any other kinds of long-term behavior? If so, what are the possibilities?

5. Consider the equation
$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 6t^2 - 1$$
   for $t > 0$.
   (a) Show that $y = t$ and $y = t^2$ are solutions of the homogeneous part.
   (b) Find the general solution of the full equation.
   (c) Find the particular solution (of the full equation) that satisfies the initial condition $y(1) = -1$, $y'(1) = 3$.

— Useful and interesting formulae —

Euler: $e^{\pm ix} = \cos(x) \pm i\sin(x)$
Variation of Parameters: $y_p(t) = c_1(t)y_1 + c_2(t)y_2$ where

$$c_1' = \frac{-y_2 f(t)}{|W(t)|}, \qquad c_2' = \frac{y_1 f(t)}{|W(t)|}, \qquad \text{where } W(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$