INSTRUCTIONS: Computers, calculators, books, notes, flailing monkeys, *etc.* are not permitted. Write your name, your instructor's name, and your recitation section number on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. [30 points] True or False. State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You MUST write the full word TRUE or FALSE — T/F will NOT be graded. For this question only you do NOT need to show your working or reasoning.

   (a) The linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions if A is $m \times n$ and $m < n$.

   (b) If A is invertible and λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

   (c) If $\mathbf{x}_0$ is an equilibrium of the non-linear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ and the eigenvalues of the linearized system are $\pm\sqrt{3}i$, then $\mathbf{x}_0$ must be a spiral node of the non-linear system (but it could be either stable or unstable).

   (d) A real-valued $n \times n$ matrix must have at least one real eigenvalue when n is odd.

   (e) It is possible to have an $n \times n$ matrix A such that the linear system $\mathbf{x}' = A\mathbf{x}$ has no equilibria.

   (f) If $AX = XB$ and $\det(X) \neq 0$, then $A = B$.

2. [25 points] A sensitive gyroscope is mounted inside the nosecone of an aircraft. The relative displacement $u(t)$ of the gyroscope with respect to its equilibrium position on the mount satisfies the differential equation

$$m\ddot{u} + 2\dot{u} + 3u = 0$$

   where the mass m of the gyroscope is easily changed by adding or removing small weights.

   Solve the following important design problem: find a condition on the mass that will keep the gyroscope from oscillating.

3. [25 points] The body of a murder victim was discovered at 11:00 A.M. The medical examiner arrived at 11:30 A.M. and found the temperature of the body was 94.6°F. The temperature of the room is 70°F. One hour later, in the same room, she took the body temperature again and found that it was 93.4°F. Assuming that Newtons law of cooling holds, *i.e.* $T' = k(T_{\text{room}} - T)$, estimate the time of death, given that usual body temperature is 98.6°F. (Note: without a calculator, you will not be able to give an actual time — give your answer in terms of calculable quantities, such as "$\sqrt{12.4}$ hours before 11 A.M." or "$\sin(0.76)$ hours after 11:30". Work in time units of hours!)

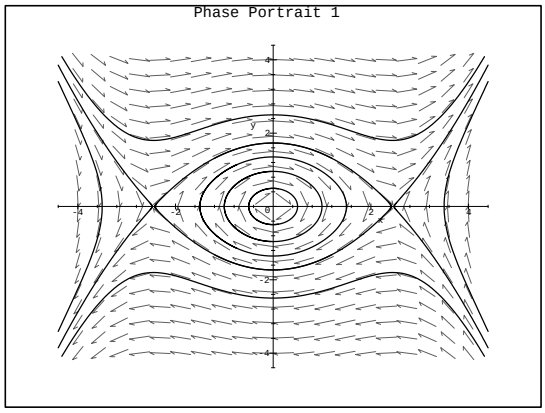
4. [25 points] Find all solutions of

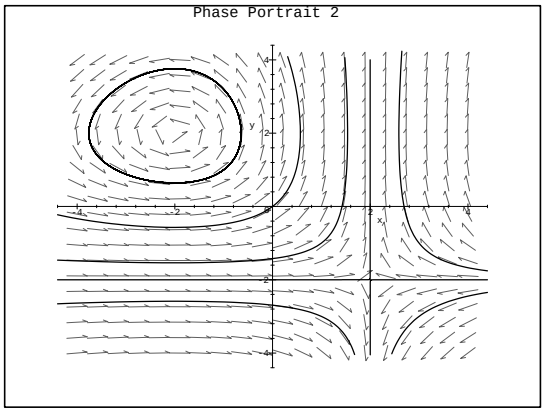
$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ 2x & + & y & - & z & = & 3 \\ x & - & y & - & 2z & = & 3 \end{array}$$

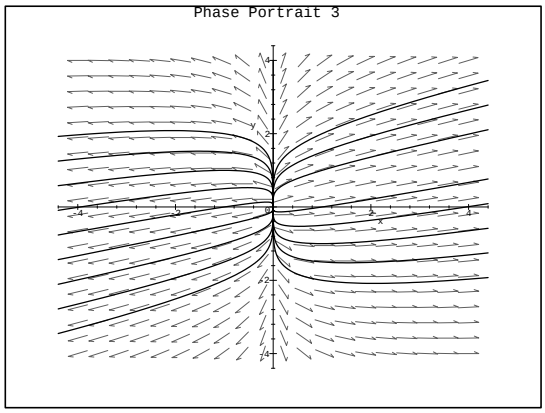
5. [15 points] Match the following differential equations with the given direction fields. It is not necessary to justify your answers for this specific problem.

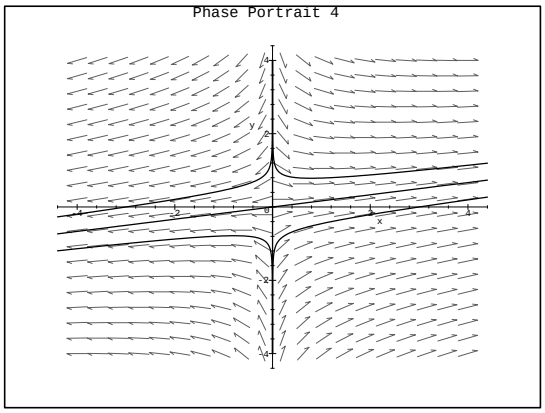
A: $\begin{cases} x' = 5x \\ y' = x - y \end{cases}$ B: $\begin{cases} x' = (x-2)(y-2) \\ y' = (x+2)(y+2) \end{cases}$ C: $\begin{cases} x' = (x+2)(y+2) \\ y' = (x-2)(y-2) \end{cases}$

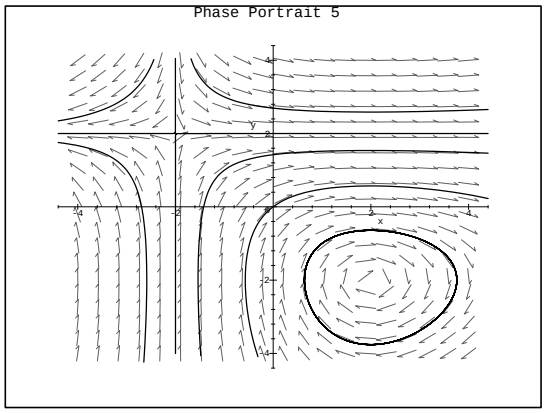
D: $\begin{cases} x' = y \\ y' = -x + x^3/6 \end{cases}$ E: $\begin{cases} x' = 5x \\ y' = x + y \end{cases}$











6. [40 points] Solve the following:

   (a) $yy' = 1 - 2t$, $y(1) = -2$

   (b) $x\frac{dy}{dx} - 3y = x^2$

   (c) $\begin{aligned} x' &= x + 2y \\ y' &= 3x + 2y \end{aligned}$

   (d) $y'' + 7y = 4y', y(0) = 0, y'(0) = 1$

7. [40 points] The following system models the interaction of a population of Spotted Cave Sloths (x) and a population of Sloth-Eating Penguins (y):

$$\begin{aligned} x' &= 4x - 2xy - x^2 \\ y' &= xy - 2y \end{aligned}$$

   Since the variables represent population sizes (measured in thousands), we need consider only $x \geq 0$ and $y \geq 0$.

   (a) Sketch the phase plane for this system. Be sure to show the equilibria and nullclines on your graph and to classify the equilibria. Use appropriate (analytic) theory to justify your classifications.

   (b) Briefly describe (in words) what happens with this system, assuming we start with at least a few Penguins and at least a few Sloths.

— Useful and interesting formulae —

Euler: $e^{\pm ix} = \cos(x) \pm i\sin(x)$
Variation of Parameters: $y_p(t) = c_1(t)y_1 + c_2(t)y_2$ where

$$c_1' = \frac{-y_2 f(t)}{|W(t)|}, \qquad c_2' = \frac{y_1 f(t)}{|W(t)|}, \qquad \text{where } W(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$