1 a) TRUE b) TRUE c) FALSE d) TRUE e) FALSE f) FALSE

2. $m\ddot{u}+2\dot{u}+3u=0 \Rightarrow u=e^{rt}$ where $mr^2+2r+3=0$

$\Rightarrow r=\frac{-2\pm\sqrt{4-12m}}{2m}$

Oscillation occurs when r is complex $\Rightarrow$ need $4-12m\geq 0 \iff \boxed{m\leq \frac{1}{3}}$

3. $t=0 \iff 11{:}30 \Rightarrow T(0)=94.6$

$T'=k(70-T) \Rightarrow \int \frac{dT}{T-70}=\int -k\,dt \Rightarrow \ln|T-70|=-kt+C$

$\Rightarrow |T-70|=e^{-kt+C}$

$\Rightarrow T=70+Ae^{-kt}$ $\quad T(0)=70+A=94.6 \Rightarrow A=24.6$

$\Rightarrow \underline{T=70+24.6e^{-kt}}$

$T(1)=70+24.6e^{-k}=93.4 \Rightarrow 23.4=24.6e^{-k} \Rightarrow e^{k}=\frac{24.6}{23.4}$

$\Rightarrow \underline{k=\ln\left(\frac{24.6}{23.4}\right)}$

Need to find t_d such that $T(t_d)=70+24.6e^{-kt_d}=98.6$

$\Rightarrow 24.6e^{-kt_d}=28.6 \Rightarrow \frac{12.3}{14.3}=e^{kt_d} \Rightarrow t_d=\frac{1}{k}\ln\left(\frac{12.3}{14.3}\right)$

$\Rightarrow \boxed{t_d=\frac{\ln\left(\frac{12.3}{14.3}\right)}{\ln\left(\frac{24.6}{23.4}\right)}=\frac{\ln(12.3)-\ln(14.3)}{\ln(24.6)-\ln(23.4)}}$

this is a negative number, so $\boxed{\text{time of death is } \left|\frac{\ln\left(\frac{12.3}{14.3}\right)}{\ln\left(\frac{24.6}{23.4}\right)}\right| \text{ hours before } 11{:}30}$

4. $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 3 \\ 1 & -1 & -2 & 3 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -3 & -3 & 3 \end{array}\right) \xrightarrow[R_2/-3]{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ z is arbitrary - let $z = t$ then $y + z = -1 \Rightarrow y = -1 - z = -1 - t$

& $x + 2y + z = 0 \Rightarrow x = -z - 2y = -t - 2(-1-t) = -t + 2t + 2 = 2 + t$

$\Rightarrow \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$

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5. $A = 4 \quad B = 2 \quad C = 5 \quad D = 1 \quad E = 3$

6a) $\int y\,dy = \int 1 - 2t\,dt \Rightarrow y^2 = 2(t - t^2) + C \Rightarrow y = \pm\sqrt{2t(1-t) + C}$

$y(0) = \pm\sqrt{C} = -2 \Rightarrow C = 4$ & take the negative square root.

$\Rightarrow \boxed{y(t) = -\sqrt{2t(1-t) + 4}}$

b) $y' - \frac{3}{x} y = x$ Int. factor $\mu = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x}$

$\Rightarrow \frac{1}{x^3} y' - \frac{3}{x^4} y = \frac{1}{x^2} \Rightarrow \left(\frac{1}{x^3} y\right)' = \frac{1}{x^2}$

$\Rightarrow \frac{1}{x^3} y = \int \frac{dx}{x^2} = \frac{-1}{x} + C$

$\Rightarrow \boxed{y = x^2(Cx - 1)}$

c) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ Eigenvalues of $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$: $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4, -1$

$\lambda = 4$: $\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right) \Rightarrow \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda = -1$: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right) \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} 2c_1 e^{4t} + c_2 e^{-t} \\ 3c_1 e^{4t} - c_2 e^{-t} \end{pmatrix}}$

6 d) $y'' - 4y' + 7y = 0$ Try $y = e^{rt}$ $\Rightarrow r^2 - 4r + 7 = 0$

$\Rightarrow r = \frac{4 \pm \sqrt{16-28}}{2} = 2 \pm \sqrt{4-7} = 2 \pm \sqrt{3}\, i$

$\Rightarrow y(t) = e^{2t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right)$

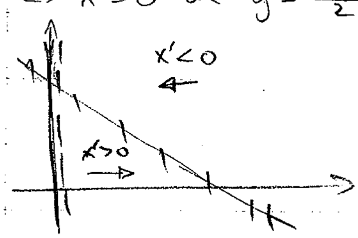
$\Rightarrow y(0) = C_1 = 0 \Rightarrow y(t) = C_2 e^{2t}\sin(\sqrt{3}t) \Rightarrow y' = C_2 e^{2t}(2\sin\sqrt{3}t + \sqrt{3}\cos\sqrt{3}t)$

$\Rightarrow y'(0) = C_2\sqrt{3} = 1 \Rightarrow C_2 = \frac{1}{\sqrt{3}} \Rightarrow \boxed{y(t) = \frac{1}{\sqrt{3}} e^{2t}\sin(\sqrt{3}t)}$

7a) Nullclines

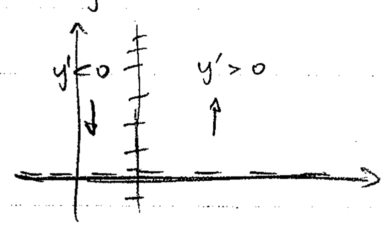
$x' = 0 \Leftrightarrow x(4-2y-x) = 0$

$\Leftrightarrow x = 0$ OR $y = \frac{4-x}{2}$



$y' = 0 \Leftrightarrow y(x-2) = 0$

$\Leftrightarrow y = 0$ OR $x = 2$

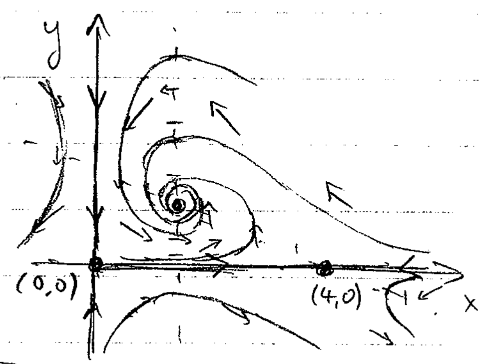


Equilibria: $x = 0$ & $y = 0 \Rightarrow (0,0)$

$x = 0$ & $x = 2$ $\not\rightarrow$

$y = \frac{4-x}{2}$ & $y = 0 \Rightarrow x = 4 \Rightarrow (4,0)$

$y = \frac{4-x}{2}$ & $x = 2 \Rightarrow y = 1 \Rightarrow (2,1)$



$DF = \begin{pmatrix} 4-2y-2x & -2x \\ y & x-2 \end{pmatrix}$

$DF(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$ $\lambda = -2, 4 \Rightarrow \boxed{(0,0) \text{ is SADDLE}}$

$DF(4,0) = \begin{pmatrix} -4 & -8 \\ 0 & 2 \end{pmatrix}$ $\lambda = 2, -4 \Rightarrow \boxed{(4,0) \text{ is SADDLE}}$

$DF(2,1) = \begin{pmatrix} -2 & -4 \\ 1 & 0 \end{pmatrix}$ $\lambda^2 + 2\lambda + 4 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}\, i \Rightarrow \boxed{(2,1) \text{ is STABLE SPIRAL}}$

b) Regardless of the initial conditions, we end up spiralling in to the equilibrium state of 2000 sloths & 1000 penguins. The spiral is an ever-decreasing oscillation of too many penguins & sloths $\rightarrow$ fewer sloths $\rightarrow$ fewer penguins $\rightarrow$ more sloths $\rightarrow$