INSTRUCTIONS: Computers, calculators, books, notes, flailing monkeys, *etc.* are not permitted. Write your name, your instructor's name, and your recitation section number on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. [30 points] True or False. State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You MUST write the full word TRUE or FALSE — T/F will NOT be graded. For this question only you do NOT need to show your working or reasoning.

   (a) The linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions if A is $m \times n$ and $n < m$.

   (b) If $A = B^{-1}CB$ (where B is some invertible matrix) then $\det(A) = \det(C)$ even though A is not necessarily equal to C.

   (c) If $\mathbf{x}_0$ is an equilibrium of the 3×3 non-linear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ and the determinant of the Jacobian matrix evaluated at $\mathbf{x}_0$ is 7, then $\mathbf{x}_0$ must be an unstable equilibrium of the non-linear system. Note: you may accept as true that the determinant of a matrix is equal to the product of its eigenvalues.

   (d) If $\mathcal{L}$ is a linear operator, u is a solution of $\mathcal{L}[y] = f(t)$, v is a solution of $\mathcal{L}[y] = g(t)$ and w is a solution of $\mathcal{L}[y] = 0$, then the general solution of $\mathcal{L}[y] = f(t) + \pi g(t)$ is $y = C_1(u + \pi v + C_2 w)$ (where C_1 and C_2 are arbitrary constants).

   (e) If A, B and $A + B$ are all invertible matrices, then $(A + B)^{-1} = B^{-1} + A^{-1}$.

2. [25 points] Dr. Prentice, Dr. Bortz, and Dr. Tearle have decided to build a flux capacitor to travel back in time to decide what color the dinosaurs really were. Their model for the machine describes the flux (x) and capacitance (y).

$$\begin{aligned} x' &= \alpha x - 3y \\ y' &= 3x - 2y \end{aligned}$$

   In order for the device to work correctly, the flux and capacitance need to oscillate with exponentially increasing amplitudes. Help them to settle this important dinosaur plumage question; what value(s) of the parameter α will make their flux capacitor work correctly?

3. [25 points] A 2-gallon tank is initially full of a salt solution at a concentration of 1lb/gal. Salt solution at a concentration of 3lb/gal is pumped into the tank at an ever-decreasing flow rate given by $r(t) = \frac{1}{1+t}$ gal/min; the well-mixed solution is drained from the tank at the same rate.

   (a) Find the amount of salt in the tank as a function of time.

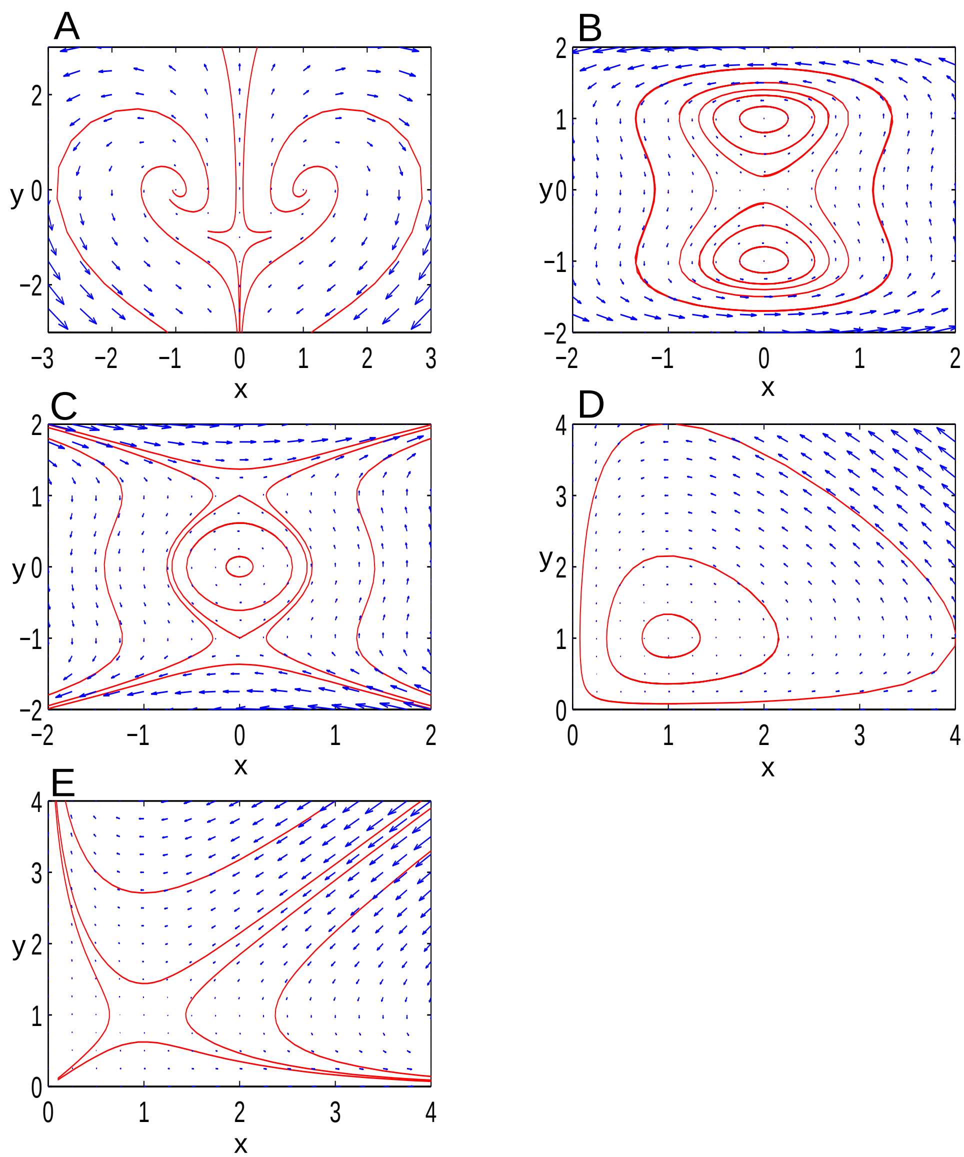
   (b) What is the long-term behavior of the <u>concentration</u> of salt in the tank? Does this make sense? Why/why not?

4. [15 points] Match the following differential equations with the given direction fields. It is not necessary to justify your answers for this specific problem.

(i) $\begin{cases} x' = xy \\ y' = y - x^2 + 1 \end{cases}$ (iv) $\begin{cases} x' = y - y^3 \\ y' = x \end{cases}$

(ii) $\begin{cases} x' = x - xy \\ y' = y - xy \end{cases}$ (v) $\begin{cases} x' = x - xy \\ y' = -y + xy \end{cases}$

(iii) $\begin{cases} x' = -y + y^3 \\ y' = x \end{cases}$



5. [25 points] For which value(s), if any, of λ does the system of equations

$$\begin{aligned} \lambda x + z &= 3 \\ 2x - 3y &= 4 \\ x - y + z &= 2 \end{aligned}$$

   (a) have a unique solution?

   (b) have no solutions?

   (c) have infinitely many solutions?

   Full credit given only for correct justification!

6. [40 points] Solve the following:

   (a) $x' = 2te^{t^2-x}$, $x(0) = \ln(3)$

   (b) $y' = y(\ln(y) - 1)$ using the substitution $z = \ln(y)$

   (c) $\begin{aligned} x' &= x + 2y \\ y' &= 3x + 2y \end{aligned}$

7. [40 points] The following system models the interaction of a population of Spotted Cave Sloths (x) and a population of Reticulated Jumping Penguins (y).

$$\begin{aligned} x' &= 4x - 2x^2 - 2xy \\ y' &= 3y - 2y^2 - xy \end{aligned}$$

   Both species live on the (in)famous island of Suluclac where rare Eulerian Rock Moss — their only source of food — grows. Since the supply of moss is limited, the Sloths and the Penguins are in competition, although they do not directly interact. Since the variables represent population sizes (measured in thousands), we need consider only $x \geq 0$ and $y \geq 0$.

   (a) Sketch the phase plane for this system. Be sure to show the equilibria and nullclines on your graph and to classify the equilibria. Use appropriate (analytic) theory to justify your classifications.

   (b) Briefly describe (in words) what happens with this system, assuming we start with at least a few Penguins and at least a few Sloths.

— Useful and interesting formulae —

Euler: $e^{\pm ix} = \cos(x) \pm i\sin(x)$

Variation of Parameters: $y_p(t) = c_1(t)y_1 + c_2(t)y_2$ where

$$c_1' = \frac{-y_2 f(t)}{|W(t)|}, \qquad c_2' = \frac{y_1 f(t)}{|W(t)|}, \qquad \text{where } W(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$