1 a) FALSE b) TRUE c) TRUE d) FALSE e) FALSE

2. $\vec{x}' = \begin{pmatrix} \alpha & -3 \\ 3 & -2 \end{pmatrix} \vec{x}$

$$\lambda^2 - (\alpha-2)\lambda + [-2\alpha+9] = 0$$

$$\Rightarrow \lambda = \frac{\alpha - 2 \pm \sqrt{(\alpha-2)^2 - 4(9-2\alpha)}}{2}$$

Need λ complex with positive real part

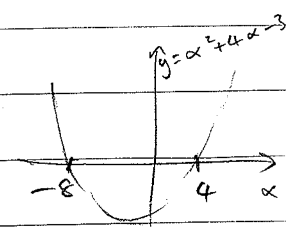
$\Rightarrow \alpha - 2 > 0$ & $(\alpha-2)^2 - 4(9-2\alpha) < 0$

$\Rightarrow \alpha > 2$ & $\alpha^2 - 4\alpha + 4 - 36 + 8\alpha < 0$

$\Leftrightarrow \alpha^2 + 4\alpha - 32 < 0$

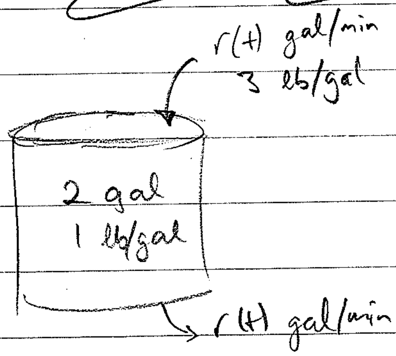
$\Leftrightarrow (\alpha+8)(\alpha-4) < 0$

$\Leftrightarrow -8 < \alpha < 4$



$\Rightarrow \boxed{2 < \alpha < 4}$

3.



$x(0) = (2\,\text{gal}) \times (1\,\text{lb/gal}) = 2\,\text{lb}.$

$$\frac{dx}{dt} = 3r(t) - \frac{x}{V} r(t) \qquad V = 2$$

$$\Rightarrow \frac{dx}{dt} = \left(3 - \frac{x}{2}\right)\frac{1}{1+t} = \frac{6-x}{2(1+t)}$$

$$\Rightarrow \int \frac{dx}{6-x} = \int \frac{dt}{2(1+t)} \Rightarrow -\ln|6-x| = \frac{1}{2}\ln|1+t| + C$$

$$\Rightarrow \ln|6-x| = \ln|(1+t)^{-1/2}| + C \Rightarrow 6 - x = A(1+t)^{-1/2} \Rightarrow \boxed{x = 6 - \frac{A}{\sqrt{1+t}}}$$

a) $\Rightarrow x(0) = 6 - A = 2 \Rightarrow A = 4 \Rightarrow \boxed{x(t) = 6 - \frac{4}{\sqrt{1+t}}}$

b) As $t \to \infty$ $x(t) \to 6$ $\Rightarrow$ concentration $\to \frac{6}{V} = \frac{6}{2} = 3 =$ concentration in which makes sense.

4 (i) A (ii) E (iii) C (iv) B (v) D

5. $$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ \lambda & 0 & \lambda & 3 \\ 2 & -3 & 0 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & \lambda & 1-\lambda & 3-2\lambda \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & \lambda & 1-\lambda & 3-2\lambda \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1-3\lambda & 3-2\lambda \end{array}\right)$$

a) If $1-3\lambda \neq 0$ then $z = \frac{3-2\lambda}{1-3\lambda}$ $\Rightarrow$ $y = -2z = \dots$

$\Rightarrow$ $x = 2+y-z = \dots$

So unique sol^{n} when $\boxed{\lambda \neq 1/3}$

b) If $\lambda = 1/3$ then system reduces to $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 7/3 \end{array}\right)$

which is inconsistent, so no sol^{n} when $\boxed{\lambda = 1/3}$

c) $\lambda = 1/3$ & $\lambda \neq 1/3$ are the only options, so $\boxed{\text{no values}}$ of λ give infinitely many sol^{n}s

6 a) $x' = 2te^{t^2}e^{-x}$ $\Rightarrow \int e^x\,dx = \int 2te^{t^2}\,dt$

$\Rightarrow e^x = e^{t^2} + C$

$\Rightarrow x = \ln(C + e^{t^2})$

$x(0) = \ln(C + e^0) = \ln(C+1) = \ln(3) \Rightarrow C = 2$

$\Rightarrow \boxed{x(t) = \ln(2 + e^{t^2})}$

b) $y' = y(\ln(y) - 1)$ $z = \ln(y) \Rightarrow \frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dt} = \frac{1}{y}\frac{dy}{dt}$

$= \cancel{\frac{1}{y}}\,\cancel{y}(\ln(y) - 1)$

$\Rightarrow z' = \ln(y) - 1 = z - 1$

$\Rightarrow \int \frac{dz}{z-1} = \int dt \Rightarrow \ln|z-1| = t + C \Rightarrow z - 1 = Ae^t$

$\Rightarrow z = 1 + Ae^t$

$\Rightarrow y = e^z = e^{1+Ae^t}$ $\boxed{y(t) = e^{1+Ae^t}}$

c) $\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}$ e-val's: $\lambda^2 - 3\lambda + (2-6) = \lambda^2 - 3\lambda - 4$

$= (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4, -1$

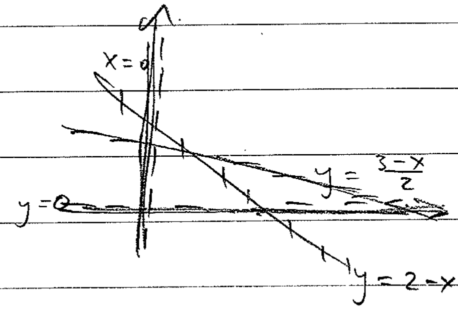
$\lambda = 4$: $\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right) \Rightarrow 3v_1 = 2v_2 \Rightarrow \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda = -1$: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right) \Rightarrow v_1 = -v_2 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}}$

7. $x' = 2x(2-x-y)$

$y' = y(3-2y-x)$



$x' = 0 \iff x = 0$ OR $y = 2-x$

$y' = 0 \iff y = 0$ OR $y = \frac{3-x}{2}$

Equilib: $x = 0 \Rightarrow y = 0$ OR $y = 3/2$

$y = 0 \Rightarrow x = 0$ OR $x = 2$

$y = 2-x$ & $y = \frac{3-x}{2} \Rightarrow 4-2x = 3-x \Rightarrow 1 = x$

$\Rightarrow y = 2-1 = 1$

$\Rightarrow \boxed{(0,0),\ (0,3/2),\ (2,0),\ (1,1)}$

Jacobian: $DF = \begin{pmatrix} 4-4x-2y & -2x \\ -y & 3-4y-x \end{pmatrix}$

$DF(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ $\lambda = 3, 4 \Rightarrow$ <u>SOURCE</u>

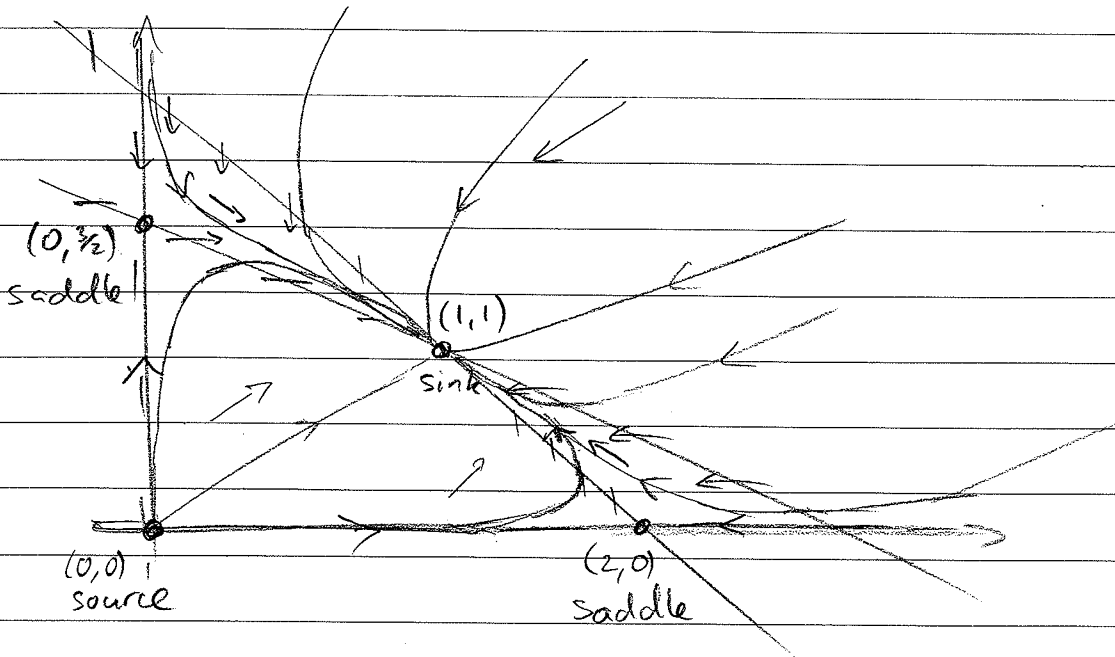
$DF(0,3/2) = \begin{pmatrix} 1 & 0 \\ -3/2 & -3 \end{pmatrix}$ $\lambda = 1, -3 \Rightarrow$ <u>SADDLE</u>

$DF(2,0) = \begin{pmatrix} -4 & -4 \\ 0 & 1 \end{pmatrix}$ $\lambda = 1, -4 \Rightarrow$ <u>SADDLE</u>

$DF(1,1) = \begin{pmatrix} -2 & -2 \\ -1 & -2 \end{pmatrix}$ $\lambda^2 + 4\lambda + 2 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}$

$\lambda_1, \lambda_2 < 0 \Rightarrow$ <u>SINK</u>

a)



b) No matter where we start (in the first quadrant) we end up at the equilibrium (1,1) as $t \to \infty$

$\Rightarrow$ We end up with a stable population of 1000 penguins & 1000 sloths co-existing peacefully in harmony with Gaia the Earth Mother, man.