APPM 2360 **FINAL EXAM** May 7, 2007

INSTRUCTIONS: This final is closed book/closed notes. Write your name, your instructor's name, and your recitation section number on the front of your bluebook. Point totals for each problem are in brackets. Work all problems. Start each problem on a **new page**. Show your work clearly and $\boxed{\text{box}}$ your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

This final contains 7 wonderful problems. Good luck! And, thank you for taking this course.☺

— Useful and interesting formulae —

- Euler: $e^{\pm ix} = \cos(x) \pm i\sin(x)$
- $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.
- Variation of Parameters: $y_p(t) = v_1(t)y_1 + v_2(t)y_2$ where

$$v_1' = \frac{-y_2 f(t)}{W(t)}, \qquad v_2' = \frac{y_1 f(t)}{W(t)}, \qquad \text{where } W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

1. [28 points] True or False. State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You MUST write the full word TRUE or FALSE — T/F will NOT be graded. For this question only you do NOT need to show your working or reasoning.

   (a) If A is a 3×3 matrix with 3 linearly independent eigenvectors, then A also has 3 distinct eigenvalues.

   (b) The determinant of a matrix with linearly independent columns is greater than zero.

   (c) If $AX = XB$ and $\det(X) \neq 0$, then $A \neq B$.

   (d) A mass-spring system for which $\ddot{x} + x = 2$ is a conservative system.

   (e) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x}$ has a center at the origin.

   (f) The functions $\{1, t-1, t^2-1\}$ are linearly independent.

   (g) Let $x_1(t)$ and $x_2(t)$ be two solutions to $x'' - 2x' + sin(t) = 0$, then $x(t) = c_1x_1(t) + c_2x_2(t)$ is also a solution.

2. [30 points] Solve the following DE's:

   (a) $x' + \frac{x}{t} = \cos(t^2)$, where $t > 0$;

   (b) $z' - \tan(t)z = 0$, where $z > 0$; and

   (c) $\omega'' - 4\omega' + 3\omega = 8te^{-t}$.

3. [25 points] A 900 gallon tank initially contains 100 gallons of water with 99 grams of kool-aid powder dissolved in the water. Water enters the tank at a rate of 24 gal/hr and the water entering the tank has a kool-aid concentration of 1 gram/gal. If a well mixed solution leaves the tank at a rate of 8 gal/hr, how much kool-aid powder is dissolved in the tank when it overflows?

4. [30 points] Consider a matrix-vector equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 1 & \alpha & -2 & 2 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & \alpha & -2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}.$$

   (a) [10 points] For what value(s) of α, if any, are the columns of A linearly **independent**?

   (b) Let $\alpha = 2$.

      i. [10 points] Find all solutions of $A\mathbf{x} = \mathbf{0}$.

      ii. [5 points] Find the solution space, W, of $A\mathbf{x} = \mathbf{0}$.

      iii. [5 points] What is $\dim(W)$?

5. [28 points] To celebrate the end of the semester, the 2360 professors decide to go bungee jumping, i.e., leap from a tall building while connected to a big rubber band. Exactly halfway up the tower, the professors pause to observe someone else jumping. The jumper zooms past the group and then $\pi/2$ seconds later, the jumper zooms by again (this time going upward!). The jumper continues to oscillate around the halfway mark on the tower with smaller and smaller amplitude, but always taking $\pi/2$ seconds between zooms past the professors. Assume that this system is described by a standard mass-spring oscillator equation

$$m\ddot{x} + b\dot{x} + kx = 0\,, \tag{1}$$

the total mass of a person wearing a safety harness is 100kg, that we can ignore wind resistance, and that it's well known that the damping coefficient of a bungee cord is 200 Newton-seconds per meter (not really, but this makes the numbers work out nicely).

   (a) [23 points] Because your professors are kind of geeky, they try to compute the spring constant of the bungee cord. What value do they find for k?

   (b) [5 points] Dr. Bortz claims he could build a small rocket pack which sinusoidally thrusts up and down with the forcing function $3\cos(\mu t)$ Newtons. He also claims he could bungee jump and use this rocket pack (with a carefully chosen forcing frequency μ to cause oscillation with increasing amplitude. Is he right? What feature of equation (1) justifies your answer?

6. [35 points] Suppose the following system of nonlinear DE's models the interaction between a population of markhors (Himalayan goats), x, and a population of snow leopards, y:

$$x' = x(6 - 2x) - 4xy$$
$$y' = 2xy - 2y.$$

Since the variables represent population sizes, you need consider only $x \geq 0$ and $y \geq 0$.

   (a) [10 points] Briefly describe (in words) the behavior x exhibits in the absence of y? i.e., when $y = 0$. What behavior does y exhibit in the absence of x?

   (b) [20 points] Sketch the phase portrait for this system. Be sure to show the equilibria and nullclines, and to classify the equilibria. Use appropriate analytic techniques to justify your classifications.

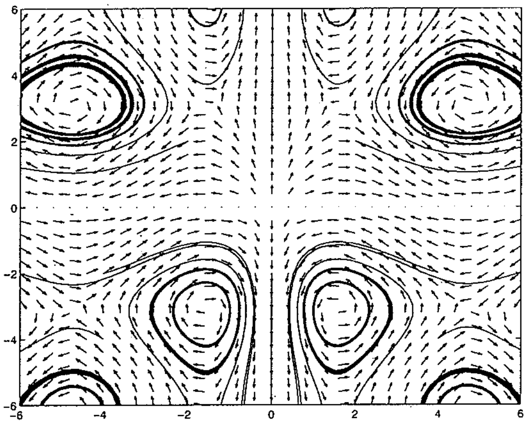
   (c) [5 points] Briefly describe (in words) what happens to the populations if there are initially at least a few markhors and at least a few snow leopards.

7. [24 points] Match each system with its phase portrait below. (You don't need to show your work.)

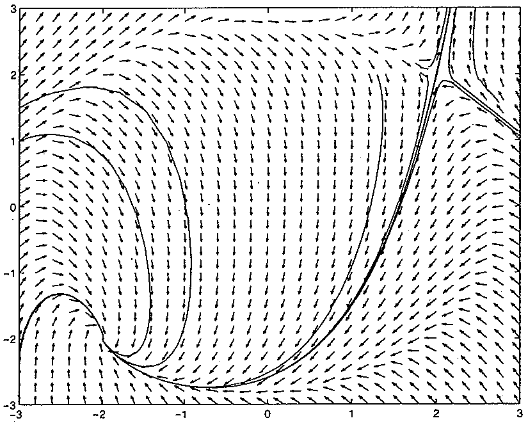
$$1.\begin{cases} x' &= xy-4 \\ y' &= y^3-4x \end{cases} \quad 2.\begin{cases} x' &= -x+5y \\ y' &= x-3y \end{cases} \quad 3.\begin{cases} x' &= y \\ y' &= -x\sin(x) \end{cases}$$

$$4.\begin{cases} x' &= y-x \\ y' &= x^2+y^2-8 \end{cases} \quad 5.\begin{cases} x' &= x+y \\ y' &= -2x-y \end{cases} \quad 6.\begin{cases} x' &= x\sin(y) \\ y' &= y\cos(x) \end{cases}$$

(A)



(B)



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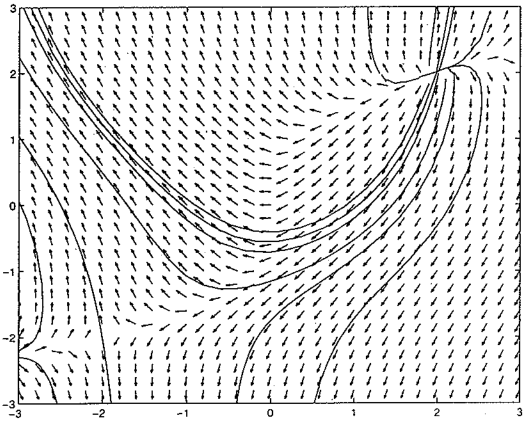

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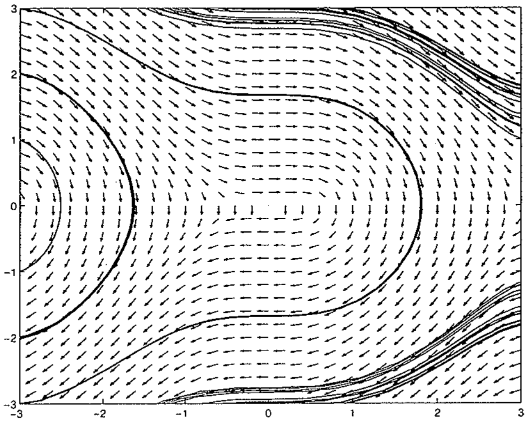
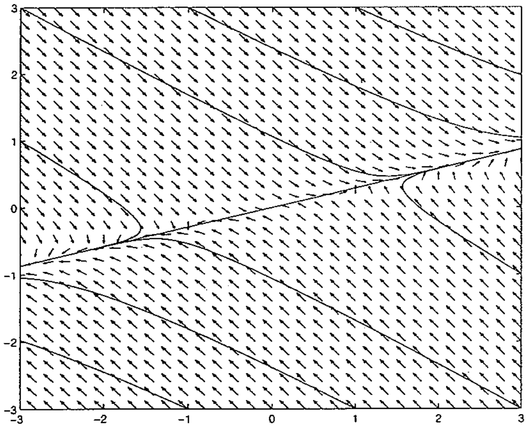


(E)



(F)