Final Exam 2360

1. a) F

   b) F

   c) F

   d) T

   e) F

   f) T

   g) F

1. a) F

   b) F

   c) F

   d) T

   e) F

   f) T

   g) F

② a) $x' + \frac{x}{t} = \cos(t^2)$, $t > 0$.

Use I.F., $\mu(t) = e^{\int \frac{1}{t} dt} = t$, to get (4)

$\Rightarrow \; t x' + x = \frac{d}{dt}(tx) = t\cos(t^2)$ (10 pts)

$tx = \frac{1}{2}\sin(t^2) + c$ (4)

$\boxed{x(t) = \frac{1}{2t}\sin(t^2) + \frac{c}{t}.}$ (2)

b) $z' - \boxed{\tan(t)}\, z = 0$, $z > 0$.

Use S.o.V. to get

$\Rightarrow \; z' = \tan(t)\, z$

$\frac{dz}{z} = \tan(t)\, dt$ (4) (10 pts)

$\ln|z| = -\ln|\cos(t)| + c_1$ (4)

$\boxed{z = \frac{c_2}{\cos(t)}}$ $c_2 = e^{c_1} > 0$ (2)

c) $w'' - 4w' + 3w = 8te^{-t}$

Homogeneous soln: $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda - 1)(\lambda - 3) = 0.$

$w_h = c_1 e^t + c_2 e^{3t}.$ (4)

Particular soln: guess $w_p = (At + B)e^{-t}$ then (10pts)

$(-2A + At + B) - 4(A - At - B) + 3(At + B) = 8t$

$A = 1,\; B = \frac{3}{4}.$ (4)

$\Rightarrow \boxed{x(t) = c_1 e^t + c_2 e^{3t} + (t + \frac{3}{4})e^{-t}}$ (2)

$$x' + \frac{x}{t} = \cos(t^2)$$

$$y_h = Ce^{-\int \frac{1}{t} dt} = Ce^{-\ln t} = \frac{c}{t}$$

$$y_p = \frac{V(t)}{t}$$

$$\frac{V't - V}{t^2} + \frac{\cancel{t} V(t)}{t^2} = \cos(t^2)$$

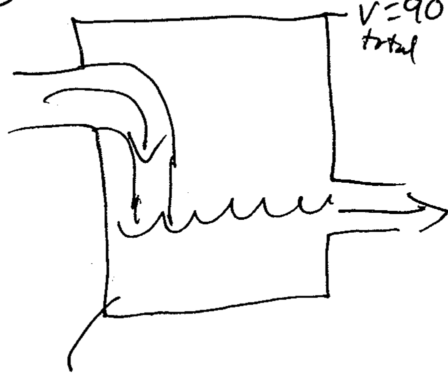
$$V' = t\cos(t^2)$$

$$V = \frac{1}{2} \cdot \sin(t^2) + C$$

$$\therefore y_p = x = \frac{\frac{1}{2}\sin(t^2) + C}{t}$$

$$\mu = e^{\int -\tan t \, dt} = e^{-(-\ln(\cos t))} = e^{\ln \cos t} = \cos(t)$$

3. [25]



$V = 900$ gal total

$V_{init} = 100$ gal

$X_0 = 99$ grams

X = grams of Kool-aid in the tank.

$$\frac{dx}{dt} = 24 \frac{\cancel{gal}}{hr} \cdot \frac{1\,gm}{\cancel{gal}} - \frac{8x}{(100+16t)}$$

[10] $\left\{ x' = 24 - \left(\frac{8}{100+16t}\right) x \right.$ —— 5

$$\mu(x) = e^{\int 8(100+16t)^{-1} dt} = (100+16t)^{1/2}$$

$$(100+16t)^{1/2} x = \int 24 (100+16t)^{1/2} dt$$

$$= \frac{24}{16} \cdot \frac{2}{3} (100+16t)^{3/2} + C$$

5

[10] $\left\{ x(t) = (100+16t) + C(100+16t)^{-1/2} \right. \longrightarrow 5$

Apply IC:

$$x(0) = 99 = 100 + \frac{1}{10} C$$

$$C = -10 \longrightarrow 5$$

General Solution:

$$900 - \frac{1}{3} = 899\frac{2}{3}$$

[-] $x(t) = 100 + 16t + -10(100+16t)^{-1/2}$

$t = 50$

$$x(50) = 100 + 16(50) + -10(100+16(50))^{-1/2}$$

$$= 100 + 800 - 10(30)$$

$= 900 - 300 = 600$ gms of Kool aid

5

4. [30 points]

(a) [10 points] All columns of A are linearly independent **iff** $\det(A) \neq 0$.

$$\begin{aligned}\det(A) &= \alpha \begin{vmatrix} \alpha & -2 & 2 \\ -1 & 1 & -1 \\ -2 & \alpha & -1 \end{vmatrix} = \alpha\left(\alpha \begin{vmatrix} 1 & -1 \\ \alpha & -2 \end{vmatrix} - (-2)\begin{vmatrix} -1 & -1 \\ -2 & -2 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ -2 & \alpha \end{vmatrix}\right) \\ &= \alpha(\alpha(-2+\alpha) + 2(-\alpha+2)) = \alpha(\alpha-2)^2.\end{aligned}$$

Answer : Any $\alpha \neq 0, 2$

(b) i. [10 points]

$$\text{RREF}: \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & 2 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -2 & 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad : \text{5 points}$$

$\Longrightarrow \quad x = 0,\ y - z + u = 0.$
Let $z = s,\ u = t$, then $y = s - t$. Therefore,

$$\mathbf{x} = \begin{pmatrix} 0 \\ s-t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{for all } s, t \in \mathbb{R}. \ : \text{5 points}$$

If one writes the solutions

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \mathbf{x} = s\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

then give the student 4 points. If one writes the solutions

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

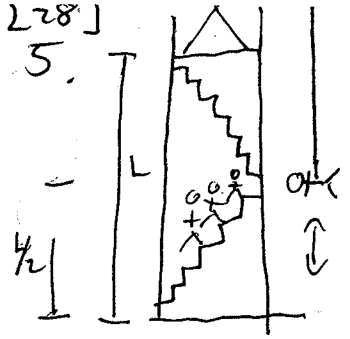
then give the student 3 points.
ii. [5 points]

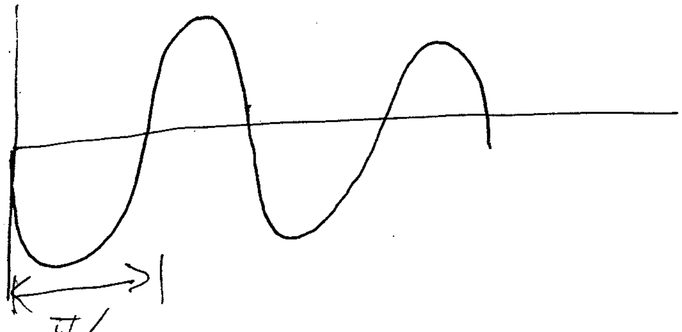
$$W = \left\{ s\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

iii. [5 points] $\dim(W) \ = \ 2$

7. [24 points] 4 points for each
1 - (C), 2 - (E), 3 - (D), 4 - (B), 5 - (F), 6 - (A)

[28]
5.





a) 8 [~~10~~]

Period of oscillations = π seconds

Natural Frequency = $\frac{1}{\pi} = \frac{1}{\pi}$ $\frac{\text{cycles}}{\text{second}}$

Angular Frequency = $\frac{2\pi}{\pi}$ = 2 $\frac{\text{radians}}{\text{second}}$

11 [10]

Equation: $100x'' + 200x' + kx = 0$

Propose $x(t) = e^{rt}$

$\rightarrow$ Char. Eqn. $100r^2 + 200r + k = 0$

$$r = \frac{-200 \pm \sqrt{40000 - 4 \cdot 100 \cdot k}}{200}$$

$$= -1 \pm \sqrt{1 - \frac{k}{100}}$$

7 [~~8~~]

So, k must be such that

$$\sqrt{1 - \frac{k}{100}} = 2i$$

$$1 - \frac{k}{100} = -4$$

$$\frac{k}{100} = 5$$

$$\boxed{k = 500}$$

b) No, there is damping in this equation, so the oscillations will not increase without bound.

[5] [2] answer [3] justification

+3 if $x \nearrow 3$ or $\searrow 3$

+3 if linear or just decay.

6 $\begin{cases} x' = x(6-2x) - 4xy \\ y' = 2xy - 2y \end{cases}$

a) In the absence of y, x obeys the logistic equation. In the absence of x, y simply has exponential decay. (10 pts)

b) x-null: $x=0$, $y = \frac{3}{2} - \frac{1}{2}x$

y-null: $y=0$, $x=1$

eq. @ $(0,0)$; $(1,1)$; $(3,0)$. (2 pts) +1 if more or less

$$J_{(x^*,y^*)} = \begin{pmatrix} 6-4x^*-4y^* & -4x^* \\ 2y^* & 2x^*-2 \end{pmatrix}$$ (2 pts)

near $(0,0)$: $J_{(0,0)} = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \lambda_1 = 6, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda_2 = -2, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\therefore$ saddle (2 pts)

near $(1,1)$: $J_{(1,1)} = \begin{pmatrix} -2 & -4 \\ 2 & 0 \end{pmatrix} \Rightarrow Tr(J) = -2$, $|A| = 8$

$8 > \frac{1}{4}(-2)^2$

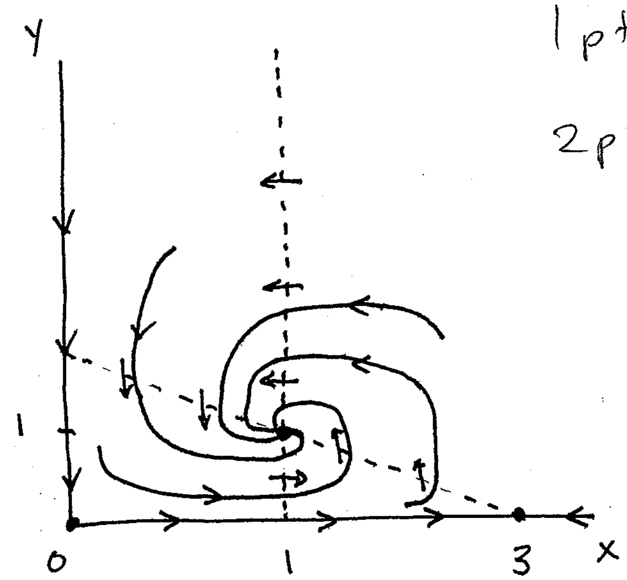
$\therefore$ spiral sink (2 pts)

near $(3,0)$: $J_{(3,0)} = \begin{pmatrix} -6 & -12 \\ 0 & 4 \end{pmatrix} \Rightarrow \lambda_1 = -6, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 4, \vec{v}_2 = \begin{pmatrix} -12 \\ 10 \end{pmatrix}$

$\therefore$ saddle (2 pts)

1 pt for each nullcline
1 pt for direction o each
2 pts for trajectories

10 pts



c) If there are initially at least a few x and y, the solution will always go to the (1,1) equilibrium as it is a stable spiral sink. This means that the populations will coexist forever.

(5 pts)