Exam #1, APPM 2360

Fall 2007

(1) (a) This is a separable equation, thus

$$\frac{dy}{dt} = 2\sqrt{y} \Rightarrow \frac{dy}{2\sqrt{y}} = dt \Rightarrow \int \frac{dy}{2\sqrt{y}} = \int dt \Rightarrow$$

$$\sqrt{y} = t + C \Rightarrow y = (t+C)^2 \quad //$$

The additional solution is $\sqrt{y} = 0 \Rightarrow y = 0 \quad //$

It is <u>not</u> included in the general solution since the equation is nonlinear.

(b) Given $y(0) = y_0 \Rightarrow y_0 = (0+C)^2 \Rightarrow y_0 = C^2$

Hence if $y_0 < 0$ solutions do not exist //

if $y_0 \geq 0$ solutions exist

Using Picard's theorem

$f(y) = 2\sqrt{y}$, thus $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$

in the area around $y(0) = y_0$ we have unique solutions if $y_0 \neq 0$.

Thus unique solutions when $y_0 > 0$ //

(2) (a) Substitute in the equation

$$\frac{dy_p}{dt} + y_p = 2\sin t \Rightarrow \frac{d}{dt}(A\cdot\sin t + B\cos t)$$

$$+ (A\sin t + B\cos t) = 2\sin t \Rightarrow$$

$$A\cos t - B\sin t + A\sin t + B\cos t = 2\sin t \Rightarrow$$

$$(A+B)\cos t + (A-B)\sin t = 2\sin t$$

Hence $\left\{\begin{matrix} A+B=0 \\ A-B=2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} A=-B \\ 2A=2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} A=1 \\ B=-1 \end{matrix}\right.$ //

(b) Since we know a particular solution we only need to solve $\frac{dy_h}{dt} + y_h = 0 \Rightarrow y_h = c e^{-t}$

The general solution is

$$y = y_h + y_p = c e^{-t} + \sin t - \cos t$$ //

(3) (a) In order for the operator to be linear

$$\begin{cases} L[cy] = c\,L[y] \\ L[y_1 + y_2] = L[y_1] + L[y_2] \end{cases}$$

However $L[cy] = \frac{d}{dt}(cy) + (cy)^2 = c\frac{dy}{dt} + c^2 y^2$

$$c\,L[y] = c \cdot \left(\frac{dy}{dt} + y^2\right) = c\frac{dy}{dt} + cy^2$$

Thus in general $L[cy] \neq c\,L[y]$

the operator is not a linear operator //

(b) $L[y] = 0 \Rightarrow y' + y^2 = 0$

Substitute $y = \frac{1}{t} + \frac{1}{u} \Rightarrow \frac{dy}{dt} = -\frac{1}{t^2} - \frac{u'}{u^2}$

Hence : $-\frac{1}{t^2} - \frac{u'}{u^2} + \left(\frac{1}{t} + \frac{1}{u}\right)^2 = 0 \Rightarrow$

$$-\cancel{\frac{1}{t^2}} - \frac{u'}{u^2} + \cancel{\frac{1}{t^2}} + \frac{1}{u^2} + \frac{2}{tu} = 0 \Rightarrow$$

$$-u' + 1 + \frac{2}{t} u = 0 \quad \Rightarrow$$

$$u' - \frac{2}{t} \cdot u = 1. \quad \text{Thus} \quad M = \frac{d}{dt} - \frac{2}{t}, \quad f(t) = 1$$

(c) $$M[cy] = \frac{d}{dt}(cy) - \frac{2}{t}(cy) = c\frac{dy}{dt} - c\frac{2}{t}y$$

$$= c\left(\frac{dy}{dt} - \frac{2}{t}y\right) = c\,M[y]$$

$$M[y_1 + y_2] = \frac{d}{dt}(y_1 + y_2) - \frac{2}{t}(y_1 + y_2)$$

$$= \frac{dy_1}{dt} + \frac{dy_2}{dt} - \frac{2}{t}y_1 - \frac{2}{t}y_2$$

$$= \left(\frac{dy_1}{dt} - \frac{2}{t}y_1\right) + \left(\frac{dy_2}{dt} - \frac{2}{t}y_2\right)$$

$$= M[y_1] + M[y_2]$$

Hence this is a linear operator.

(4) (a) The mass is given to be

$$m(t) = \delta \cdot V = \delta \cdot \frac{4\pi}{3} r^3 = 1 \cdot \frac{4\pi}{3}(kt)^3 \Rightarrow$$

$$m(t) = \frac{4\pi}{3} k^3 t^3 \quad //$$

(b) Substitute in the equation

$$\frac{d}{dt}\left[\frac{4\pi}{3} k^3 t^3 \cdot v\right] = \frac{4\pi}{3} k^3 t^3 \cdot g \Rightarrow$$

$$\cancel{\frac{4\pi}{3}} \cancel{k^3}\left(3t^2 \cdot v + t^3 v'\right) = \cancel{\frac{4\pi}{3}} \cancel{k^3} t^3 g \Rightarrow$$

$$v' + \frac{3}{t} v = g$$

Integrating factor is $\mu = e^{\int P dt} = e^{\int \frac{3}{t} dt} = t^3$

Thus: $\mu \cdot v = \int Q \cdot \mu \, dt + C_1 \Rightarrow$

$$t^3 v = \int g t^3 dt + C_1 \Rightarrow$$

$$t^3 v = \frac{g}{4} t^4 + C_1 \Rightarrow$$

$$v = \frac{g}{4} t + \frac{C}{t^3}$$

For the initial condition to be satisfied

$$v(0) = 0 \Rightarrow C = 0$$

Finally $\quad v(t) = \frac{g}{4} t \quad //$

Alternativelly: Integrate the equation

$$\frac{d}{dt}[m \cdot v] = m \cdot g \Rightarrow m \cdot v = \int m \cdot g \, dt + C \Rightarrow$$

$$\frac{4\pi}{3} k^3 t^3 v = \frac{4\pi}{3} k^3 g \int t^3 dt + C_1 \Rightarrow t^3 v = g \frac{t^4}{4} + C_1$$

$$\Rightarrow \quad v = \frac{g}{4} t + \frac{C}{t^3} \quad //$$

(c) Since $\quad v = \frac{g}{4} t, \quad \frac{dv}{dt} = g/4 \quad //$

(5) (a) Field (A) : Homogeneous //

Field (B) : Non-homogeneous //

(b) Field (C) : Non linear //

(c) Field (A) : $y_2 = ce^{-2t}$

Field (B) : $y_3 = ce^{-2t} + \frac{t}{2} - \frac{1}{4}$

Field (C) : $y_1 = \frac{e^t}{e^t - c}$