Exam #2, APPM 2360

Fall 2007

(1) (a) False, take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$

however $|A| = 1 - 1 = 0$.

(b) False, if A is $n \times m$ and B $m \times n$ then

AB is $n \times n$ and its determinant is defined

However, the determinants for A and B are not!

(c) False, one of the equations may be redundant

(d) False, take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $|A| = -1 \neq 0$

(e) False, take $A\ (n \times m)$, $B\ (i \times j)$ then

$\underset{(n\times m)}{A}\ \underset{(i\times j)}{B}\ \underset{(n\times m)}{A}$ thus we need $m = i$ or $j = n$

$A\ (n \times m)\ B\ (m \times n)$ and ABA is well defined.

(2) (a) If $x = \cos A$, $y = \cos B$, $z = \cos C$ the system is

$$\begin{cases} 0 \cdot x + c \cdot y + b \cdot z = a \\ c \cdot x + 0 \cdot y + a \cdot z = b \\ b \cdot x + a \cdot y + 0 \cdot z = c \end{cases}$$

Hence $A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in

$A \vec{u} = \vec{v}$

(b) $|A| = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = -c \begin{vmatrix} c & a \\ b & 0 \end{vmatrix} + b \begin{vmatrix} c & 0 \\ b & a \end{vmatrix}$

$= c \cdot ab + b \cdot ca = 2abc$

Since a, b, c are sides of a triangle $|A| \neq 0$ and the system has a unique solution

(c) If $a=b=c$, $|A| = 2a^3$

$$|A_1| = \begin{vmatrix} a & a & a \\ a & 0 & a \\ a & a & 0 \end{vmatrix} = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} =$$

$$a^3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - a^3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + a^3 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -a^3 + a^3 + a^3 = a^3$$

$$|A_2| = \begin{vmatrix} 0 & a & a \\ a & a & a \\ a & a & 0 \end{vmatrix} = a^3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -a^3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + a^3 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = a^3$$

$$|A_3| = \begin{vmatrix} 0 & a & a \\ a & 0 & a \\ a & a & a \end{vmatrix} = a^3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= -a^3 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + a^3 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = a^3$$

Finally: $x = \dfrac{|A_1|}{|A|} = \dfrac{a^3}{2a^3} = 1/2$ //

$$y = \frac{|A_2|}{|A|} = \frac{a^3}{2a^3} = 1/2 \text{ //}$$

$$z = \frac{|A_3|}{|A|} = \frac{a^3}{2a^3} = 1/2 \text{ //}$$

hence $x = y = z = 1/2$. As expected

if in a triangle all its sides are equal

then all its angles are also equal

and so is $\cos A = \cos B = \cos C$. //

(3) (a) $\vec{u}(0) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$, $\vec{v}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Then $\begin{vmatrix} -1 & 2 \\ -4 & 0 \end{vmatrix} = 8 \neq 0$ hence $\vec{u}(0)$ and $\vec{v}(0)$ are linerly independent//

(b) $\vec{w}(0) = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$ and we need to find c_1 and c_2 such that $c_1 \cdot \vec{u}(0) + c_2 \vec{v}(0) = \vec{w}(0)$

$$\Rightarrow c_1 \begin{bmatrix} -1 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \Rightarrow$$

$$\left\{ \begin{array}{l} -c_1 + 2c_2 = 6 \\ -4c_1 + 0c_2 = 12 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = -3 \\ c_2 = \dfrac{6 + c_1}{2} = 3/2 \end{array} \right.$$

Finally: $\vec{w}(0) = -3\,\vec{u}(0) + \frac{3}{2}\,\vec{v}(0)$ //

(c) $A'(x) = \frac{dA}{dx} = \frac{d}{dx}\begin{bmatrix} x^2-1 & 2 \\ x-4 & x^2-3x \end{bmatrix}$

$$= \begin{bmatrix} 2x & 0 \\ 1 & 2x-3 \end{bmatrix}$$

$\mathrm{col}(A') = \left\{ \begin{bmatrix} 2x \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2x-3 \end{bmatrix} \right\}$, hence

$$|A'| = \begin{vmatrix} 2x & 0 \\ 1 & 2x-3 \end{vmatrix} = 0 \Rightarrow 2x(2x-3) = 0$$

Hence col (A') spans $\mathbb{R}^2$ for all x

<u>except</u> $x=0$ and $x = 3/2$ //

(4) (a) Form the Wronskian determinant

$$W = \begin{vmatrix} 1+x & -x & 1-x^2 & x^3 \\ 1 & -1 & -2x & 3x^2 \\ 0 & 0 & -2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1+x & -x & 1-x^2 \\ 1 & -1 & -2x \\ 0 & 0 & -2 \end{vmatrix} =$$

$$= -12 \begin{vmatrix} 1+x & -x \\ 1 & -1 \end{vmatrix} = -12 \left[-1 - x + x \right] =$$

$$= -12 \cdot (-1) = 12 \neq 0$$

for all x they are linearly independent.
Since the number of vectors match the dimension of the space they form a basis //

(b) Now $P_3(x) = a_1 x + a_2 x^2 + a_3 x^3$

(i) $P_3^{(1)}(x) + P_3^{(2)}(x) = a_1 x + a_2 x^2 + a_3 x^3$

$+ b_1 x + b_2 x^2 + b_3 x^3$

$= (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$

$= c_1 x + c_2 x^2 + c_3 x^3 \in \mathbb{P}^3$

for all values of c_1, c_2 and c_3

(ii) $c\,P_3(x) = c \cdot a_1 x + c a_2 x^2 + c a_3 x^3$

$= d_1 x + d_2 x^2 + d_3 x^3 \in \mathbb{P}^3$

for all values of d_1, d_2, d_3

thus the polynomials with $P_3(0) = 0$

form a subspace of $\mathbb{P}^3$

(c) (i) Now a_0 is not zero necessarily thus

$$P_3^{(1)}(x) + P_3^{(2)}(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$$
$$+ (a_3+b_3)x^3$$

If $a_3+b_3=0$ then the polynomial $P_3^{(1)}(x) + P_3^{(2)}(x)$ is second order and thus does not be long in $\mathbb{P}^3$.

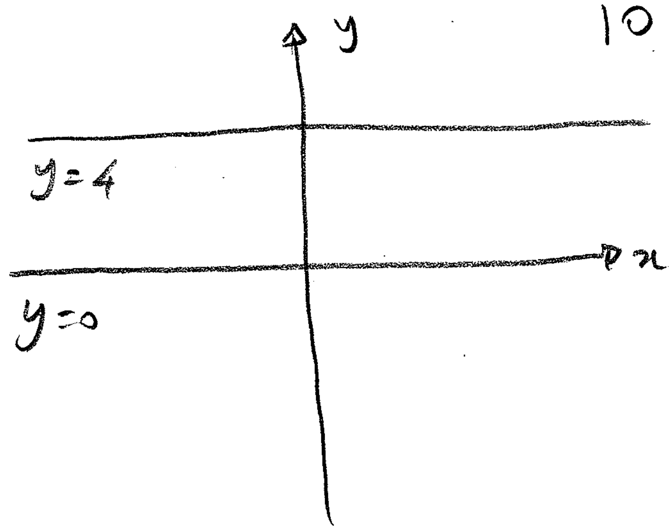
Hence they <u>don't</u> form a subspace for $\mathbb{P}^3$

(5) (a) Equilibria are found when $\frac{dy}{dx} = y(4-y)-h=0$

Thus $y(4-y)-h = -y^2+4y-h=0$

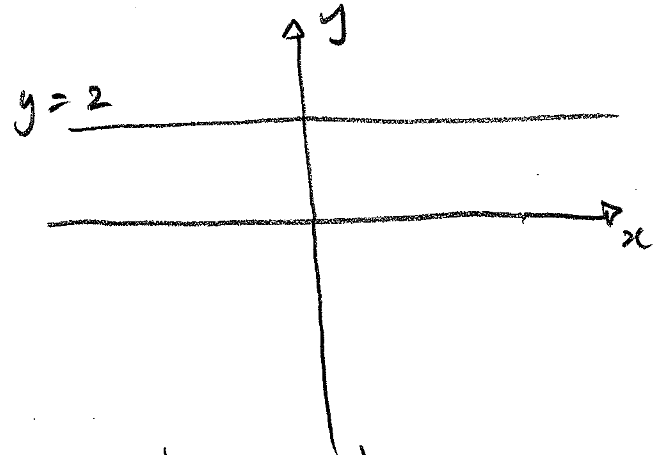
$\Delta = 16-4h$, in order to have two distinct, real equilibria $\Delta > 0 \Rightarrow 16-4h > 0 \Rightarrow h<4$

(b) $\underline{h=0}$: $\dfrac{dy}{dx} = y(4-y)$



$\underline{h=4}$: $\dfrac{dy}{dx} = -y^2+4y-4$

$= -(y-2)^2$



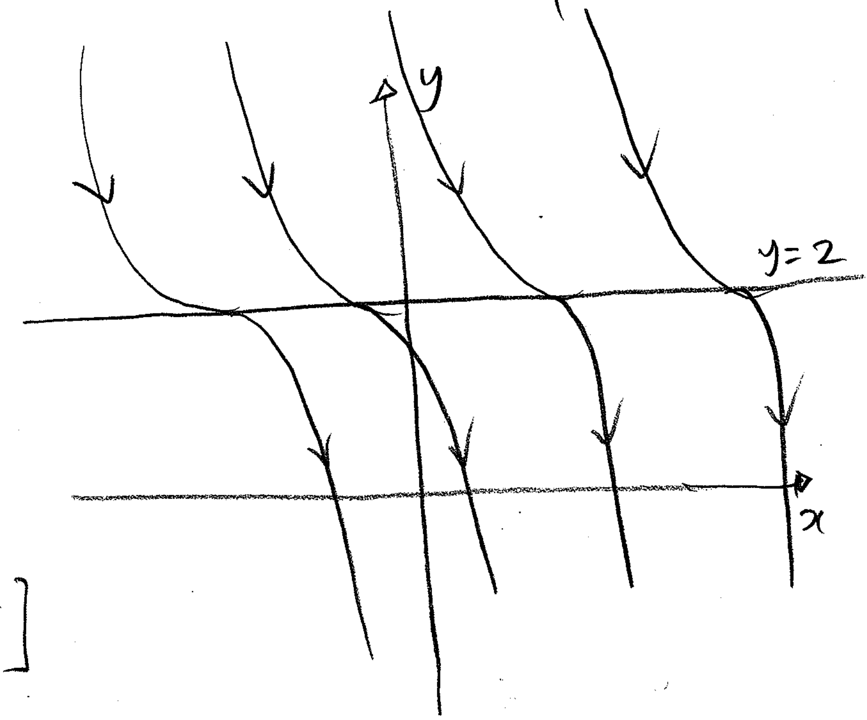
(c) $\dfrac{dy}{dx} = \underbrace{-(y-2)^2}_{f(y)} < 0$

$\dfrac{d^2y}{dx^2} = \dfrac{df}{dy}\cdot\dfrac{dy}{dx} = \dfrac{df}{dy}\cdot f$

$= -2(y-2)\cdot\left[-(y-2)^2\right]$

$= 2(y-2)^3$

$\begin{cases} y''>0 & \text{if } y>2 \\ y''<0 & \text{if } y<2 \end{cases}$



hence as $x \to +\infty$ this is a <u>stable</u> equilibrium //

Extra credit: $A^7 = A^3 \cdot A^3 \cdot A = \begin{bmatrix} 8 & -7 \\ 7 & -6 \end{bmatrix}$

then $A^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ with $(A^3)^{-1} = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

since $\det(A^3) = -8+9 = 1$. Hence

$$A^3 A = (A^3)^{-1} \begin{bmatrix} 8 & -7 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & -7 \\ 7 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$$

and finally $A = (A^3)^{-1} \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix} =$

$$= \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \quad //$$