Exam # 3, APPM 2360

Fall 2007

(1) (a) $t^2 y_1'' - 2t y_1' + 2y_1 = t^2 (t)'' - 2t(t)' + 2t$

$$= 0 - 2t + 2t = 0$$

Hence $y_1 = t$ is a solution. //

(b) Take $y = t \cdot y_2$, then

$$y' = y_2 + t y_2' \quad , \quad y'' = 2y_2' + t y_2''$$

Substitute

$$t^2 (2y_2' + t y_2'') - 2t (y_2 + t y_2') + 2t y_2 = 0 \Rightarrow$$

$$\cancel{2t^2 y_2'} + t^3 y_2'' - \cancel{2t y_2} - \cancel{2t^2 y_2'} + \cancel{2t y_2} = 0 \Rightarrow$$

$$t^3 y_2'' = 0 \Rightarrow y_2'' = 0 \Rightarrow y_2 = c_1 t + c_2$$

Finally $y = t y_2 = c_1 t^2 + c_2 t$, $y_2 = t^2$ //

(c) $\left[t^2 y'' - 2ty' + 2y \right]' = 0 \Rightarrow$

$\cancel{2ty''} + t^2 y''' - \cancel{2y'} - \cancel{2ty''} + \cancel{2y'} = 0 \Rightarrow \quad t^3 y''' = 0 \Rightarrow$

$y''' = 0$ //

The solution is $y = c_1 t^2 + c_2 t + c_3$, hence y_1 and y_2 are included in the solution space. //

(d) $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{dy}{dx} \cdot \dfrac{d\ln t}{dt} = \dfrac{1}{t}\dfrac{dy}{dx}$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{1}{t}\frac{dy}{dx}\right] = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2}\frac{dx}{dt}$$

$$= \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}$$

Substitute : $t^2\left[\dfrac{1}{t^2}\dfrac{d^2y}{dx^2} - \dfrac{1}{t^2}\dfrac{dy}{dx}\right]$

$- 2t\left[\dfrac{1}{t}\dfrac{dy}{dx}\right] + 2y = 0 \Rightarrow$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2\frac{dy}{dx} + 2y = 0 \Rightarrow$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \quad //$$

Hence $A_1 = 1, \quad A_2 = -3, \quad A_3 = 2 \quad //$

To solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ we assume

$$y = e^{rx} = e^{r\ln t} = (e^{\ln t})^r = t^r \quad //$$

(2)(a) $w = y_1 y_2' - y_1' y_2 \Rightarrow w' = (y_1 y_2')' - (y_1' y_2)'$

$$= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} =$$

$$= y_1 y_2'' - y_1'' y_2 \quad //$$

(b) $W' = y_1 y_2'' - y_1'' y_2$

$$= y_1\left[-Py_2' - Qy_2\right] - y_2\left[-Py_1' - Qy_1\right]$$

$$= -Py_1y_2' - Q\cancel{y_1y_2} + Py_2y_1' + Q\cancel{y_2y_1}$$

$$= -P\left[y_1y_2' - y_1'y_2\right] = -PW \quad //$$

(c) Since $P(t) = t$, $W' = -tW \Rightarrow$

$$\frac{dw}{dt} = -tw \Rightarrow \frac{dw}{w} = -t\,dt \Rightarrow \ln W = -t^2/2 + C \Rightarrow$$

$W = e^{-t^2/2} \cdot C$ when $t = 0$ $w(0) = C \neq 0$

hence $w(t) \neq 0$ for all t and thus the equation has two linearly independent solutions

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(3) (a) Here $\Delta = b^2 - 4mk < 0$ and

$x = e^{rt}$ where $r_1, r_2 = \dfrac{-b \pm i\sqrt{4mk - b^2}}{2m}$

Thus $p = \dfrac{b}{2m}$ and $\omega = \dfrac{\sqrt{4mk - b^2}}{2m}$ //

(b) Minima or maxima occur at $x'(t) = 0 \Rightarrow$

$A\left[e^{-pt}\cos(\omega t - \delta)\right]' = 0 \Rightarrow$

$-pe^{-pt}\cos(\omega t - \delta) - e^{-pt}\cdot\omega\sin(\omega t - \delta) = 0 \Rightarrow$

$-p\cos(\omega t - \delta) = \omega\sin(\omega t - \delta) \Rightarrow \tan(\omega t - \delta) = -\dfrac{p}{\omega}$ //

(c) Two consecutive maxima occur say at t_1 and t_2, hence $x_1 = x(t_1)$, $x_2 = x(t_2)$ where t_1 and t_2 satisfy $\tan(\omega t - \delta) = -\frac{\rho}{\omega}$

Also $\omega t_2 - \omega t_1 = 2\pi \Rightarrow t_2 = t_1 + \frac{2\pi}{\omega}$

$$x_1 = A e^{-\rho t_1} \cos(\omega t_1 - \delta)$$

$$x_2 = A e^{-\rho t_2} \cos(\omega t_2 - \delta) = A e^{-\rho(t_1 + \frac{2\pi}{\omega})} \cos(\omega t_1 - \delta + 2\pi)$$

$$= A e^{-\rho t_1} e^{-\frac{2\pi\rho}{\omega}} \cos(\omega t_1 - \delta)$$

$$= x_1 e^{-2\pi\rho/\omega} \Rightarrow$$

$$\frac{x_2}{x_1} = e^{-2\pi\rho/\omega} \Rightarrow \frac{x_1}{x_2} = e^{2\pi\rho/\omega} \Rightarrow$$

$$\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\rho}{\omega} \quad //$$

(d) The general solution is going to be

$$x = x_h + x_p = A e^{-pt} \cos(\omega t - \delta) + x_p$$

where $\quad x_p = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

$$x_p' = -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t)$$

$$x_p'' = -\omega^2 c_1 \cos(\omega t) - \omega^2 c_2 \sin(\omega t)$$

$$m \ddot{x}_p + b \dot{x}_p + k x_p = E_0 \cos(\omega t) + F_0 \sin(\omega t)$$

$$-m\omega^2 c_1 \cos(\omega t) - m\omega^2 c_2 \sin(\omega t) - b\omega c_1 \sin(\omega t)$$

$$+ b\omega c_2 \cos(\omega t) + k c_1 \cos(\omega t) + k c_2 \sin(\omega t) =$$

$$= E_0 \cos(\omega t) + F_0 \sin(\omega t) \Rightarrow$$

$$\left.\begin{cases} -m\omega^2 c_1 + b\omega \cdot c_2 + k c_1 = E_0 \\ -m\omega^2 c_2 - b\omega c_1 + k c_2 = F_0 \end{cases}\right\} \Rightarrow$$

$$\begin{cases} (k - m\omega^2)\, c_1 + (b\omega)\, c_2 = E_0 \\ (-b\omega)\, c_1 + (k - m\omega^2)\, c_2 = F_0 \end{cases}$$

$$A = \begin{bmatrix} k - m\omega^2 & b\omega \\ -b\omega & k - m\omega^2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} E_0 & b\omega \\ F_0 & k - m\omega^2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} k - m\omega^2 & E_0 \\ -b\omega & F_0 \end{bmatrix}, \quad |A| = (k - m\omega^2)^2 + b^2\omega^2 \neq 0$$

$$c_1 = \frac{|A_1|}{|A|} = \frac{E_0 (k - m\omega^2) - F_0 b\omega}{(k - m\omega^2)^2 + b^2\omega^2} \quad //$$

$$c_2 = \frac{|A_2|}{|A|} = \frac{F_0 (k - m\omega^2) + E_0 b\omega}{(k - m\omega^2)^2 + b^2\omega^2} \quad //$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -4 \\ 1 \end{bmatrix} \Rightarrow$$

$$\left\{ \begin{array}{l} x_1 = 4c_1 e^{t} - 4c_2 e^{-t} \\ x_2 = c_1 e^{t} + c_2 e^{-t} \end{array} \right\} \text{ Also}$$

//

$$\left\{ \begin{array}{l} x_1(0) = 4c_1 - 4c_2 = x_0 \\ x_2(0) = c_1 + c_2 = x_0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 - c_2 = x_0/4 \\ c_1 + c_2 = x_0 \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{l} c_1 = 5x_0/8 \;// \\ c_2 = x_0 - c_1 = x_0 - \frac{5x_0}{8} = \frac{3x_0}{8} \;// \end{array} \right.$$

(c) $$\left\{ \begin{array}{l} \dot{x}_1 = 4x_2 \\ \dot{x}_2 = \frac{1}{4} x_1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \ddot{x}_1 = 4\dot{x}_2 = 4\left(\frac{1}{4} x_1\right) \\ \ddot{x}_2 = \frac{1}{4} \dot{x}_1 = \frac{1}{4}(4x_2) \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{l} \ddot{x}_1 = x_1 \\ \ddot{x}_2 = x_2 \end{array} \right. \;//$$ and hence $\ddot{x}_1(0) = \dot{x}_2(0) = x_0$ //

(5) (a) → (v) , (d) → (ii)

(b) → (iv) , (e) → (i)

(c) → (iii)

Extra credit : $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn}-\lambda \end{vmatrix} = 0$

Expand the determinant using the cofactors of $(a_{ii} - \lambda)$ every time The highest order term in $c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0 = 0$ is the product of the coefficients of λ hence $c_n = (-1)^n$

The two expressions are the same for all values of λ and hence for $\lambda = 0$, thus

$$|A| = c_0$$