ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

1. (*20 points, 4 each*) State whether the following statements are *always* "TRUE" or "FALSE" (meaning not *always* true). You MUST write the full word TRUE or FALSE. T/F or YES/NO will NOT be given any credit.

(a) If a 2×2 system $\mathbf{x}' = A\mathbf{x}$ has eigenvalues $\lambda_1 = -1, \lambda_2 = (-1 + i)$ then the origin is a stable equilibrium point.

(b) The initial value problem $y' = \sqrt{y}$ with initial condition $y(0) = 1$ admits a unique solution.

(c) If A is an $n \times n$ matrix such that $A^2 = A$ then either $|A| = 0$ or $|A| = 1$.

(d) The functions e^{at} and e^{bt} are linearly independent if and only if $a \neq b$.

(e) If A, P, D are arbitrary square matrices of the same dimension, and $A = PDP^{-1}$, where P^{-1} is the inverse of P, then $A^n = PD^nP^{-1}$ for any positive integer n.

2. (*16 points, 8 each*) Consider the linear system of differential equations

$$\mathbf{x}' = A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{x}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the solution of the system that corresponds to initial conditions $\mathbf{x}(0) = [1, 0, 1]^T$.

3. (*24 points, 8 each*) Consider the second order linear differential equation

$$(x^2 + 1)y'' - 2xy' + 2y = 2 \tag{1}$$

(a) Verify that $y_1(x) = x$ and $y_2(x) = x^2 - 1$ are two linearly independent solutions of the homogeneous equation.

(b) Find a particular solution of Eq. (1).

(c) Find the solution of Eq. (1) that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

4. (*24 points, 8 each*) Consider the system

$$\begin{aligned} \lambda\, x + (\lambda + 2)y &= 2 \\ x + \lambda\, y &= 1 \end{aligned}$$

where λ is a real constant.

(a) Find the values of λ for which the system does *not* have a unique solution.

(b) Find the solution (x_0, y_0) of the system, for the values of λ for which the system has a unique solution.

(c) Find the value of λ such that the system has a unique solution (x_0, y_0) that also minimizes the function $f(\lambda) = x_0 + 3y_0^2$.

5. (*16 points, 8 each*) Consider the initial value problem

$$\frac{dy}{dt} = \frac{y\cos t}{1+2y^2} \quad y(0) = 1$$

(a) Solve the equation and apply the initial condition to find the solution of the initial value problem.

(b) Prove that the solution satisfies the inequality: $0 \leqslant \ln|y| + y^2 \leqslant 2$.

6. (*24 points, 8 each*) Consider a car of mass m and assume that the suspension system acts like a single spring and its shock absorbers like a single dashpot. The vertical vibrations of the car then satisfy

$$mx'' + cx' + kx = 0\,,$$

where m, c, k are positive constants.

(a) In the case of a *critically* damped system the solution of the equation is given by

$$x(t) = (A + Bt + Apt)e^{-pt} \tag{2}$$

Specify p in terms of k, c, m and A, B in terms of the initial conditions $x_0 = x(0)$ and $v_0 = x'(0)$.

(b) Show using the solution (2) that the mass passes through $x = 0$ at some instant $t > 0$ if and only if x_0 and $v_0 + px_0$ have opposite signs.

(c) Show using the solution (2) that $x(t)$ has a single local maximum or minimum at some instant $t > 0$ if and only if v_0 and $v_0 + px_0$ have the same sign.

7. (*16 points, 8 each*) Let **V** be the set of all infinite sequences $\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, \ldots, s_n, \ldots\}$, where the s_n's are real numbers. Consider the subset **W** consisting of those elements $\{s_n\}$ such that

$$s_{\ell+1} = s_\ell + s_{\ell-1}$$

for $\ell \geqslant 1$.

(a) Show that **W** is a subspace of **V**.

(b) Prove that **W** is 2-dimensional.

8. (*24 points, 8 each*) Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= y + hx(x^2+y^2) \\ \frac{dy}{dt} &= -x + hy(x^2+y^2) \end{aligned}$$

where h is a real parameter.

(a) Show that $(0,0)$ is the only equilibrium of the system.

(b) Linearize the system about the equilibrium point and find the eigenvalues of the linearized system, thus showing that $(0,0)$ is a center for the linear system obtained by setting $h = 0$.

(c) Suppose that $h \neq 0$ and $r^2 = x^2 + y^2$. Use

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

and the nonlinear system to show that $\frac{dr}{dt} = hr^3$

9. (*32 points, 8 each*) A pair of rabbits produces each month a new pair of rabbits. Each new pair is similarly productive beginning from the second succeeding month. Let s_n denote the number of rabbit pairs after n months. Consider the vector $\mathbf{x}_n$ given by

$$\mathbf{x}_n = \begin{bmatrix} s_{n+1} \\ s_n \end{bmatrix}$$

which is such that

$$\mathbf{x}_n = A\mathbf{x}_{n-1} \qquad \text{with} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

If you start with one pair the initial condition is $\mathbf{x}_0 = (1,1)^T$ and

$$\mathbf{x}_n = A^n \mathbf{x}_0 \,. \tag{3}$$

(a) Find the eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of A.

(b) Using matrix multiplication, show that

$$A = PDP^{-1} \tag{4}$$

where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $P = [\mathbf{v}_1 \ \ \mathbf{v}_2]$ and P^{-1} is the inverse of P.

(c) Use the formula $A^n = PD^nP^{-1}$ and matrix multiplication to compute A^n.
Hint: You may use that the n-th power of a diagonal matrix is simply obtained by raising each diagonal element to the same power.

(d) Use A^n from part (c) and Eq. (3) to find $\mathbf{x}_n$

$$\mathbf{x}_n \equiv \begin{bmatrix} s_{n+1} \\ s_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Extra credit: (*5 points*) Use the result for x_n from part (d) to show that

$$s_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

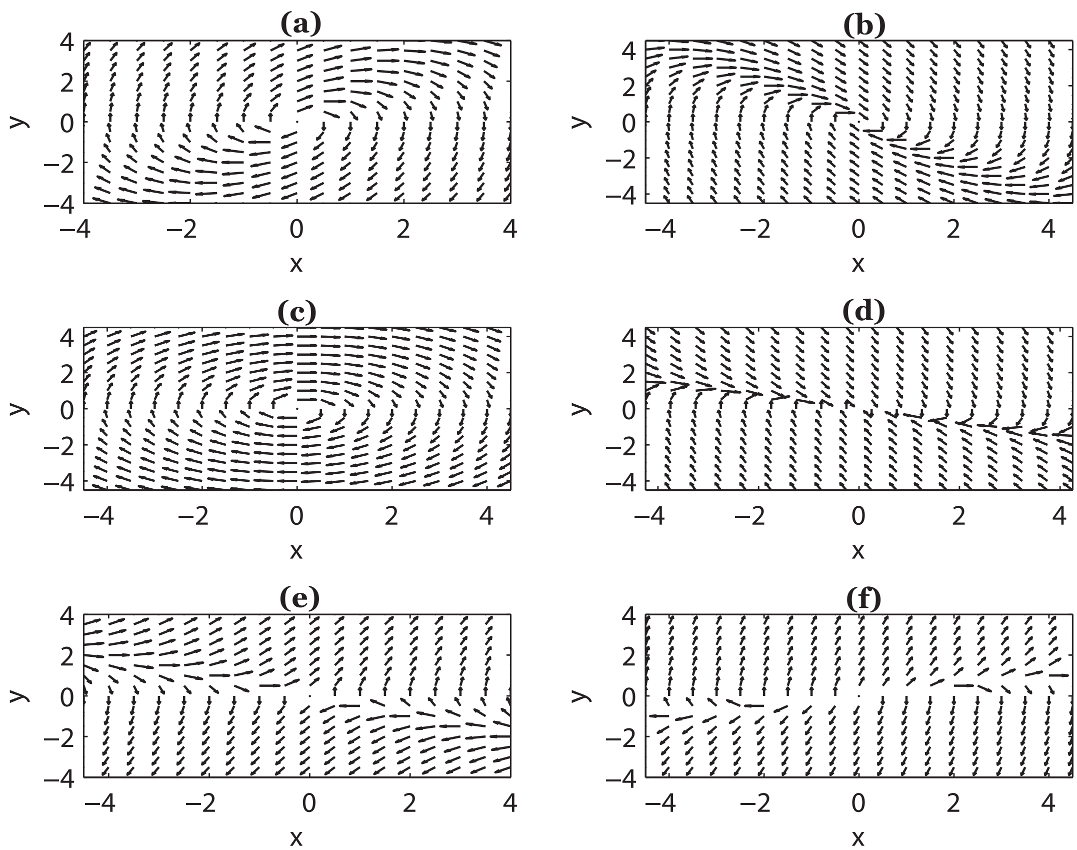
which gives the number of rabbit pairs after n months.

10. (*24 points, 4 each*) Match each linear system with the corresponding phase-plane. You do NOT have to justify your answer.

(i) $\begin{cases} x' = y \\ y' = y - 2x \end{cases}$ **(ii)** $\begin{cases} x' = y \\ y' = -2y - 2x \end{cases}$

(iii) $\begin{cases} x' = y \\ y' = -x \end{cases}$ **(iv)** $\begin{cases} x' = y \\ y' = -3y - x \end{cases}$

(v) $\begin{cases} x' = y \\ y' = 2y + x \end{cases}$ **(vi)** $\begin{cases} x' = y \\ y' = 4y - x \end{cases}$



Good Luck!!!