owo

1. 20 points, 4 each

(a) F
(b) T
(c) T
(d) T
(e) T

2. 16 points, 8 each

(a)
$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2+4\right]$$

$\lambda_1 = 1$

$\lambda_2 = 1+2i$, $\lambda_3 = 1-2i$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 = x_1 \\ 2x_1 - x_2 - 2x_3 = x_2 \\ 3x_1 + 2x_2 + \cancel{x_3} = \cancel{x_3} \end{cases} \Leftrightarrow \begin{cases} x_2 + x_3 = x_1 \\ 2x_2 = -3x_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_3 = x_1 - x_2 = x_1 + 3/2\, x_1 = 5/2\, x_1 \\ x_2 = -3/2\, x_1 \end{cases}$$

$$\vec{v}_1 = x_1/2 \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \quad \text{so} \quad \vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (1+2i)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 = (1+2i)x_1 \\ 2x_1 + x_2 - 2x_3 = (1+2i)x_2 \\ 3x_1 + 2x_2 + x_3 = (1+2i)x_3 \end{cases}$$

$$\begin{cases} x_1 = 0 \\ x_3 = -ix_2 \\ x_2 = ix_3 \end{cases} \quad \text{so} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\vec{v}_3 = \vec{v}_2^{\,*} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(c)

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + e^{t} \cdot c_2 \left\{ \cos 2t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} + e^{t} c_3 \left\{ \sin 2t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \cos 2t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so $\begin{cases} 2c_1 = 1 \\ -3c_1 + c_2 = 0 \\ 5c_1 + c_3 = 1 \end{cases} \iff \begin{array}{l} c_1 = 1/2 \\ c_2 = 3/2 \\ c_3 = 1 - 5/2 = -3/2 \end{array}$

3. (a) $y_1 = x, \quad y_1' = 1, \quad y_1'' = 0 \Rightarrow -2x + 2x = 0$

$y_2 = x^2 - 1, \quad y_2' = 2x, \quad y_2'' = 2 \Rightarrow 2(x^2+1) - 2x(2x) + 2(x^2-1) = 0$

$$W[y_1, y_2] = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 + 1 = x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$$

so y_1, y_2 are l.i.

(b) $y_p = 1$ is a particular solution

(c) $y(x) = c_1 x + c_2(x^2 - 1) + 1 \Rightarrow y(0) = -c_2 + 1 = 0$

$y'(x) = c_1 + 2c_2 x \Rightarrow y'(0) = c_1 = 1$

so $c_1 = c_2 = 1$

4. 24 points, 8 each

$$\begin{cases} \lambda x + (\lambda+2)y = 2 \\ x + \lambda y = 1 \end{cases} \qquad \lambda \in \mathbb{R}$$

(a) $$\begin{vmatrix} \lambda & \lambda+2 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0 \iff \lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

for $\lambda = 2$, $\lambda = -1$ the system does not admit a unique sol

(b) $$x_0 = \frac{\begin{vmatrix} 2 & \lambda+2 \\ 1 & \lambda \end{vmatrix}}{\lambda^2-\lambda-2} = \frac{2\lambda - \lambda - 2}{\lambda^2-\lambda-2} = \frac{\lambda-2}{\lambda^2-\lambda-2} = \frac{1}{\lambda+1}$$

$$y_0 = \frac{\begin{vmatrix} \lambda & 2 \\ 1 & 1 \end{vmatrix}}{\lambda^2-\lambda-2} = \frac{\lambda-2}{\lambda^2-\lambda-2} = \frac{1}{\lambda+1}$$

(c) $$f(\lambda) = \frac{1}{\lambda+1} + \frac{3}{(\lambda+1)^2} = \frac{\lambda+1+3}{(\lambda+1)^2} = \frac{\lambda+4}{(\lambda+1)^2}$$

$$f'(\lambda) = \frac{(\lambda+1)^2 - 2(\lambda+1)(\lambda+4)}{(\lambda+1)^4} = \frac{\lambda+1-2(\lambda+4)}{(\lambda+1)^3} = \frac{\lambda+1-2\lambda-8}{(\lambda+1)^3} = \frac{-\lambda-7}{(\lambda+1)^3}$$

so $f'(\lambda) = 0 \iff \boxed{\lambda = -7}$

5. 16 points, 8 each

$$\frac{dy}{dt} = \frac{y\cos t}{1+2y^2} \qquad y(0) = 1$$

(a) $y=0$ equilibrium solution

Take $y \neq 0$ and separate variables

$$\left(\frac{1}{y} + 2y\right) dy = \cos t \, dt \qquad \infty$$

$$\ln|y| + y^2 = \sin t + C$$

$$t=0 \quad y(0) = 1 \quad \Rightarrow \qquad C = 1$$

$$\ln|y| + y^2 = \sin t + 1$$

(b) $0 \leq \sin t + 1 \leq 2$ Therefore $0 \leq \ln|y| + y^2 \leq 2$

6. 24 points, 8 each

$$mx'' + cx' + kx = 0 \qquad mr^2 + cr + k = 0$$

(a) critical damping $\Rightarrow$ $\Delta = c^2 - 4km = 0$

and $r = -\dfrac{c}{2m}$, solns $x_1 = e^{-\frac{c}{2m}t}$

$x_2 = t\,e^{-\frac{c}{2m}t}$

∞ $p = c/2m$

$t=0 \qquad x(0) = A = x_0$

$$x' = (B + Ap)e^{-pt} - p e^{-pt}(A + Bt + Apt) \qquad \infty$$

$t=0 \qquad x'(0) = B + Ap - Ap \equiv B = v_0$

$$x(t) = (x_0 + v_0 t + x_0 p t) e^{-pt}$$

(b) $x(t_0) = 0 \iff x_0 + (v_0 + x_0 p) t_0 = 0 \iff$

$t_0 = -\dfrac{x_0}{v_0 + x_0 p} > 0$ only if x_0 and $v_0 + x_0 p$

have opposite sign

(c) $x'(t) = e^{-pt}\left(v_0 + \cancel{x_0 p} - p\cancel{x_0} - p v_0 t - x_0 p^2 t\right)$

$x'(t_0) = 0 \iff v_0 - p(v_0 + x_0 p) t_0 = 0$

$\iff t_0 = \dfrac{v_0}{p(v_0 + x_0 p)}$.

note $p > 0$ $\left(p = \frac{c}{2m}\right)$ so $t_0 > 0$ iff v_0 and

$v_0 + x_0 p$ have the same sign.

7. (16 points, 8 each)

(a) $\{s_n\}_{n\in\mathbb{N}}$, $\{p_n\}_{n\in\mathbb{N}}$

$\{s_n\}_{n\in\mathbb{N}} + \{p_n\}_{n\in\mathbb{N}} \equiv \{s_n + p_n\}_{n\in\mathbb{N}}$

$s_{e+1} + p_{e+1} = s_e + s_{e-1} + p_e + p_{e-1} = (s_e + p_e) + (s_{e-1} + p_{e-1})$

closed under addition

$c\, s_{e+1} = c(s_e + s_{e-1}) = c\, s_e + c\, s_{e-1}$

closed under scalar multiplication

b) W is 2 dimensional because once s_1 and s_2 are given, all the other terms in the sequence are uniquely specified

8) 24 points (8 each)

$$\frac{dx}{dt} = y + hx(x^2+y^2)$$

$$\frac{dy}{dt} = -x + hy(x^2+y^2)$$

a) equilibria $\quad y + hx(x^2+y^2) = 0 \quad \Leftrightarrow$

$$-x + hy(x^2+y^2) = 0$$

obviously $x = 0 = y$ is a solution

$$x^2+y^2 = -\frac{y}{hx} = \frac{x}{hy} \quad \Leftrightarrow \quad -y^2 = x^2 \quad \Leftrightarrow \quad x = y = 0$$

b) $$J = \begin{bmatrix} 3hx^2 + hy^2 & 1 + 2hxy \\ -1 + 2hxy & hx^2 + 3hy^2 \end{bmatrix} \qquad J\big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

eigenvalues $\quad 0 = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \quad \Rightarrow \quad \lambda = \pm i$

$\Rightarrow$ center!

Obviously the same as $h = 0$.

c) $\quad h \neq 0, \quad r^2 = x^2 + y^2$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = \cancel{xy} + hx^2(x^2+y^2) - \cancel{yx} + hy^2(x^2+y^2)$$

$$r\frac{dr}{dt} = hr^4 \quad \Leftrightarrow \quad \frac{dr}{dt} = hr^3$$

9. . 32 points, 8 each

$$X_m = \begin{bmatrix} f_{m+1} \\ f_m \end{bmatrix}$$

Then $\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = X_m = A X_{m-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f_n + f_{n-1} \\ f_n \end{bmatrix} \rightarrow$ eq - identity

(a)
$$0 = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(\frac{1+\sqrt{5}}{2}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \quad v_1 = \left(\frac{1+\sqrt{5}}{2}\right) v_2, \quad v_1 + v_2 = \left(\frac{1+\sqrt{5}}{2}\right) v_1$$

$$\vec{v}_1 = \begin{bmatrix} 1+\sqrt{5}/2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(\frac{1-\sqrt{5}}{2}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \quad v_1 + v_2 = \left(\frac{1-\sqrt{5}}{2}\right) v_1, \quad v_1 = \frac{1-\sqrt{5}}{2} v_2$$

$$\vec{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \qquad |P| = \tfrac{1}{2} + \tfrac{\sqrt{5}}{2} - \tfrac{1}{2} + \tfrac{\sqrt{5}}{2}$$

$$PDP^{-1} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$= \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^2 & \left(\frac{1-\sqrt{5}}{2}\right)^2 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \cdot \frac{1}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2 & \left(\frac{1+\sqrt{5}}{2}\right)^2 \left(\frac{-1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^2 \left(\frac{1+\sqrt{5}}{2}\right) \\ \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} & \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{-1+\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right) & \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\left[-\frac{1}{2} - \frac{\sqrt{5}}{2} + \frac{1-\sqrt{5}/2}{2}\right] \\ \sqrt{5} & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt5}\begin{bmatrix} \sqrt5 & \frac{1-5}{4}\cdot(-\sqrt5) \\ \sqrt5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{ok!}$$

Then

$$A^n = P D^n P^{-1} = \begin{bmatrix} \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\begin{bmatrix} 1 & \frac{-1+\sqrt5}{2} \\ -1 & \frac{1+\sqrt5}{2} \end{bmatrix}\frac{1}{\sqrt5}$$

$$= \frac{1}{\sqrt5}\begin{bmatrix} \lambda_1^n\left(\frac{1+\sqrt5}{2}\right) & \lambda_2^n\left(\frac{1-\sqrt5}{2}\right) \\ \lambda_1^n & \lambda_2^n \end{bmatrix}\begin{bmatrix} 1 & \frac{-1+\sqrt5}{2} \\ -1 & \frac{1+\sqrt5}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt5}\begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n\left(\frac{1+\sqrt5}{2}\right)\left(\frac{-1+\sqrt5}{2}\right) + \lambda_2^n\left(\frac{1-\sqrt5}{2}\right)\left(\frac{1+\sqrt5}{2}\right) \\ \lambda_1^n - \lambda_2^n & -\lambda_1^n\lambda_2 + \lambda_1\lambda_2^n \end{bmatrix}$$

so

$$A^n = \frac{1}{\sqrt5}\begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n & \lambda_1\lambda_2\left(\lambda_2^{n-1} - \lambda_1^{n-1}\right) \end{bmatrix}$$

$$x_n = \frac{1}{\sqrt5}\begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n & \lambda_1\lambda_2\left(\lambda_2^{n-1} - \lambda_1^{n-1}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt5}\begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} + \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n - \lambda_1^n\lambda_2 + \lambda_1\lambda_2^n \end{bmatrix} = \begin{bmatrix} \partial_{n+1} \\ \partial_n \end{bmatrix}$$

$$\lambda_1 = \frac{1+\sqrt5}{2} \qquad \lambda_2 = \frac{1-\sqrt5}{2} \qquad \lambda_1 = \lambda_2 + \sqrt5$$

$$\lambda_1^n - \lambda_2^n - \lambda_1^n(\lambda_1 - \sqrt5) + (\lambda_2 + \sqrt5)\lambda_2^n$$

$$\underline{\lambda_1^n} - \lambda_2^n - \lambda_1^{n+1} - \sqrt5\,\underline{\lambda_1^n} + \lambda_2^{n+1} + \sqrt5\,\lambda_2^n$$

$$(1-\sqrt5)\lambda_1^n - (1-\sqrt5)\lambda_2^n$$

Extra Credit

$$\lambda_1^n - \lambda_2^n - \lambda_1^n \lambda_2 + \lambda_1 \lambda_2^n = \lambda_1^n - \lambda_2^n - \lambda_1^n\left(\lambda_1 - \sqrt{5}\right) + \left(\lambda_2 + \sqrt{5}\right)\lambda_2^n$$

$$= \lambda_1^n - \lambda_2^n - \lambda_1^{n+1} + \sqrt{5}\,\lambda_1^n + \lambda_2^{n+1} + \sqrt{5}\,\lambda_2^n$$

$$= \lambda_1^n\left(1+\sqrt{5}\right) - \lambda_2^n\left(1-\sqrt{5}\right) - \lambda_1^{n+1} + \lambda_2^{n+1}$$

$$= 2\lambda_1^{n+1} - 2\lambda_2^{n+1} - \lambda_1^{n+1} + \lambda_2^{n+1}$$

$$= \lambda_1^{n+1} - \lambda_2^{n+1}$$

8 bin

$$A^n = PD^nP^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2}$$

$$= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2}$$

$$= \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & -\lambda_2 \lambda_1^{n+1} + \lambda_1 \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n & -\lambda_2 \lambda_1^n + \lambda_1 \lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2}$$

$$X_n = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} - \lambda_2 \lambda_1^{n+1} + \lambda_1 \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n - \lambda_2 \lambda_1^n + \lambda_1 \lambda_2^n \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} - \lambda_1 \lambda_2 (\lambda_1^n - \lambda_2^n) \\ \lambda_1^n - \lambda_2^n - \lambda_1 \lambda_2 (\lambda_1^{n-1} - \lambda_2^{n-1}) \end{bmatrix}$$

$\lambda_1 - \lambda_2 = \sqrt{5}$

10. (i) - (a)
(ii) - (b)
(iii) - (c)
(iv) - (d)
(v) - (e)
(vi) - (f)

4 points each