ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

1. (*20 points, 4 each*) State whether the following statements are *always* "TRUE" or "FALSE" (meaning not *always* true). You MUST write the full word TRUE or FALSE. T/F or YES/NO will NOT be given any credit.

(a) If the $n \times n$ matrix A satisfies $|A| = 0$ then $A = \mathbb{O}$, where $\mathbb{O}$ is the zero matrix.

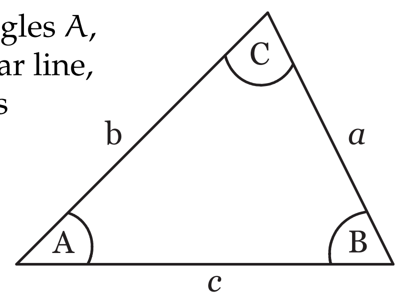
(b) For *all* matrices A and B, $|AB| = |A||B|$.

(c) A system of 3 unknowns and 4 equations can *never* have a unique solution.

(d) If the diagonal elements of the square matrix A are all zero, then $|A|$ must also be zero.

(e) If for the two matrices A and B the product ABA is well defined, then A and B are square matrices of the same dimension.

2. (*24 points, 8 each*) Consider the acute triangle on the right, with angles A, B and C and opposite sides a, b and c. By taking the perpendicular line, from each vertex to the opposite side, we can derive the equations



$$\begin{aligned} c\cos B + b\cos C &= a \\ c\cos A + a\cos C &= b \\ a\cos B + b\cos A &= c \end{aligned}$$

Regarding these equations as a system in the unknowns $x = \cos A$, $y = \cos B$ and $z = \cos C$:

(a) Write the system in the form $A\mathbf{u} = \mathbf{v}$, where $\mathbf{u} = [x \;\; y \;\; z]^T$ and clearly define the matrix A and the vector $\mathbf{v}$.

(b) Show that the system has a unique solution.

(c) If $a = b = c$, solve the system and thus prove that $x = y = z$. Without solving the system, do you expect that $x = y = z$? Explain.

3. (*24 points, 8 each*) Consider the vectors

$$\mathbf{u}(x) = \begin{bmatrix} x^2 - 1 \\ x - 4 \end{bmatrix}, \quad \mathbf{v}(x) = \begin{bmatrix} 2 \\ x^2 - 3x \end{bmatrix}, \quad \mathbf{w}(x) = \begin{bmatrix} x + 6 \\ x^3 + 12 \end{bmatrix}$$

and the matrix $A(x) = [\mathbf{u} \;\; \mathbf{v}]$, whose columns are the vectors $\mathbf{u}$ and $\mathbf{v}$.

(a) Show that the vectors $\mathbf{u}(0)$ and $\mathbf{v}(0)$ are linearly independent.

(b) Write $\mathbf{w}(0)$ as a linear combination of $\mathbf{u}(0)$ and $\mathbf{v}(0)$.

(c) Define the matrix $A'(x) = \dfrac{dA}{dx}$. For which values of x the column space of $A'(x)$ spans $\mathbb{R}^2$?

More on the back; turn over the page.

4. (*21 points, 7 each*) Consider the set $\mathbb{P}^3$ of all polynomials of degree $n \leqslant 3$, i.e. polynomials of the form $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where a_0, a_1, a_2 and a_3 are real constants.

(a) Show that the set $S = \{1 + x,\ -x,\ 1 - x^2,\ x^3\}$ forms a basis for $\mathbb{P}^3$.

(b) Show that the subset of all polynomials satisfying $P_3(0) = 0$ is a subspace of $\mathbb{P}^3$. (**Hint**: $P_3(0) = 0 \Leftrightarrow a_0 = 0$)

(c) Show that the subset of all polynomials of degree $n = 3$, i.e. polynomials of the form $Q_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, with $a_3 \neq 0$, is not a subspace of $\mathbb{P}^3$.

5. (*21 points, 7 each*) Consider the differential equation

$$\frac{dy}{dx} = y(4 - y) - h, \quad h > 0$$

which describes a logistic equation with harvesting h.

(a) Show that in order to have two distinct and real equilibria $h < 4$.

(b) Draw the equilibria when $h = 0$ and when $h = 4$.

(c) Draw the phase plane of the equation when $h = 4$. Discuss the stability of the solutions around the equilibria.

Extra credit: (*5 points*) If $A^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ and $A^7 = \begin{bmatrix} 8 & -7 \\ 7 & -6 \end{bmatrix}$ find A.

Good Luck!!!