ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

1. (*24 points, 6 each*) Consider the differential equation

$$t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0$$

(a) Verify that $y_1(t) = t$ is a solution of the equation.

(b) Find a second solution $y_2(t)$ to the equation.

(c) Differentiate the equation and show that $\frac{d^3y}{dt^3} = 0$ for $t \neq 0$.
Are y_1 and y_2 contained in the solution space of this equation? Explain.

(d) Make the substitution $x = \ln t$ to transform the equation for $y(t)$ into the constant coefficient linear equation for $y(x)$

$$A_1\frac{d^2y}{dx^2} + A_2\frac{dy}{dx} + A_3y = 0$$

and define the constants A_1, A_2 and A_3. Thus conclude that the general solution of the original equation is of the form $y = t^r$, where r is a constant.

2. (*24 points, 8 each*) Let y_1 and y_2 be two linearly independent solutions of the differential equation

$$y'' + P(t)y' + Q(t)y = 0$$

where $P(t)$ and $Q(t)$ are continuous functions.

(a) Define the Wronskian determinant $W = y_1y_2' - y_1'y_2$ of the equation and take its first derivative to show that $W' = y_1y_2'' - y_2y_1''$.

(b) Substitute for the second derivatives using the equation to show that $W' = -P(t)W$.

(c) If $P(t) = t$ and $Q(t) = 1$ find $W(t)$.

3. (*24 points, 6 each*) An underdamped oscillating system is described by the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

where m, b and k are real positive constants.

(a) Define p and ω such that $x(t) = Ae^{-pt}\cos(\omega t - \delta)$ is the general solution of the equation.

(b) Show that the local minima and maxima of $x(t)$ occur at $\tan(\omega t - \delta) = -\frac{p}{\omega}$.

(c) If x_1 and x_2 are two consecutive local maximum values of $x(t)$, then show: $\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi p}{\omega}$.
Hint: You may use that $\omega t_2 - \omega t_1 = 2\pi$.

(d) Find the general solution of the forced oscillation described by the differential equation

$$m\ddot{x} + b\dot{x} + kx = E_0\cos(\omega t) + F_0\sin(\omega t)$$

where E_0 and F_0 are constants.

More on the back; turn over the page.

4. (*18 points, 6 each*) Consider the matrix $A = \begin{bmatrix} 0 & 4 \\ 1/4 & 0 \end{bmatrix}$

(a) Find the eigenvalues and eigenvectors of A.

(b) Solve the system $\dot{\mathbf{x}} = A\mathbf{x}$, with the initial condition $\mathbf{x}(0) = [x_0, x_0]^T$. The vector $\mathbf{x}(t)$ is defined as $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ and x_0 is a constant.

(c) Take the above system in the form

$$\begin{cases} \dot{x}_1 = 4x_2 \\ \dot{x}_2 = \frac{1}{4}x_1 \end{cases}$$

Find an equation for both $\ddot{x}_1$ and $\ddot{x}_2$ and thus, or otherwise, show that $\ddot{x}_1(0) = \ddot{x}_2(0) = x_0$.

5. (*20 points, 4 each*) Match the particular solutions on the right to the differential equations on the left:

(a) $y'' - y' - 2y = 2t + 1 - 2e^t$	(i) $-2t\cos(5t)$
(b) $y'' - 2y' + y = 3e^t$	(ii) $-\frac{1}{4}\sin(5t)$
(c) $y'' - y' + 2y = 3e^t$	(iii) $\frac{3}{2}e^t$
(d) $y'' + y = 6\sin(5t)$	(iv) $\frac{3}{2}t^2e^t$
(e) $y'' + 25y = 20\sin(5t)$	(v) $-t + e^t$

Extra credit: (*5 points*) Suppose the characteristic equation of an $n \times n$ matrix A

$$|A - \lambda I| = 0$$

is written as a polynomial in the form

$$|A - \lambda I| = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0 = 0$$

where c_n, c_{n-1}, ..., c_2, c_1, c_0 are constants and I the identity matrix. Shown that $c_n = (-1)^n$ and $c_0 = |A|$.

Good Luck!!!