APPM 2360, Exam #1

Spring 2008

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(1)(a) $\frac{dy}{dt} = ty^2 \Rightarrow \frac{dy}{y^2} = t\,dt \Rightarrow \int \frac{dy}{y^2} = \int t\,dt \Rightarrow$

$-\frac{1}{y} = t^2/2 + C \Rightarrow y = -\frac{1}{C + t^2/2}$ //

The constant solution $y=0$ is not included because the equation is nonlinear //

(b) $y(t_0) = y_0 \Rightarrow -\frac{1}{C + t_0^2/2} = y_0 \Rightarrow C + t_0^2/2 = -\frac{1}{y_0} \Rightarrow$

$C = -\frac{t_0^2}{2} - \frac{1}{y_0}$ // The vertical asymptote is

$t_A^2 = -2C \Rightarrow t_A^2 = t_0^2 + \frac{2}{y_0} \Rightarrow t_A = \sqrt{t_0^2 + \frac{2}{y_0}}$ //

(c) The only solution with no vertical asymptote is $y=0$ //

(2) (a) We need to show $\frac{dy_p}{dt} \equiv 3t^2 y_p + 3t^5$

$$\frac{dy_p}{dt} = \frac{d}{dt}(-t^3-1) = -3t^2$$

$$3t^2 y_p + 3t^5 = 3t^2(-t^3-1) + 3t^5 = -3t^5 - 3t^2 + 3t^5 = -3t^2$$

Thus $\frac{dy_p}{dt} \equiv 3t^2 y_p + 3t^5$ //

(b) Since we know a particular solution and the equation is linear we only need to solve $\frac{dy_h}{dt} = 3t^2 y_h$

and the general solution will be $y = y_h + y_p$

$$\frac{dy_h}{dt} = 3t^2 y_h \Rightarrow \frac{dy_h}{y_h} = 3t^2 dt \Rightarrow \ln y_h = t^3 + C \Rightarrow$$

$y_h = ce^{t^3}$. Thus $y = ce^{t^3} - t^3 - 1$ //

(c) $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = nt^{n-1}\frac{dy}{dx}$ and substitute

$$nt^{n-1}\frac{dy}{dx} = 3t^2y + 3t^5 \Rightarrow \frac{dy}{dx} = \frac{3}{n}t^{3-n}y + \frac{3}{n}t^{6-n}$$

Compare to $\frac{dy}{dx} = y + x$ to find $n=3$ //

(d) Solve $\frac{dy}{dx} - y = x$ using integrating factor

$$y = \frac{1}{\mu}\int \mu Q\,dx + \frac{c}{\mu}, \quad \mu = e^{-\int dx} = e^{-x}, \quad Q = x$$

$$y = e^x\int e^{-x}\cdot x\,dx + e^x\cdot c = e^x\underbrace{\left(-e^{-x}\cdot x - e^{-x}\right)}_{\text{integration by parts}} + ce^x$$

$y = ce^x - x - 1$ // Then $y(x) = y(t^3)$

$$= ce^{t^3} - t^3 - 1 \;//$$

Thus the two solutions are consistent.

(3) (a) Since $u = \ln y \iff y = e^u$ thus

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = e^u \frac{du}{dt}$$, hence

$$e^u \frac{du}{dt} + e^u \cdot u - e^t e^u = 0 \Rightarrow \frac{du}{dt} + u = e^t$$

since $e^u \neq 0$. Thus $M = \frac{d}{dt} + 1$, $f(t) = e^t$ //

(b) Take $L[y_1 + y_2] = \frac{dy_1}{dt} + \frac{dy_2}{dt} + (y_1+y_2)\ln(y_1+y_2) - e^t(y_1+y_2)$

$$L[y_1] + L[y_2] = \frac{dy_1}{dt} + y_1 \ln y_1 - e^t y_1 + \frac{dy_2}{dt} + y_2 \ln y_2 - e^t y_2$$

Clearly $L[y_1+y_2] \neq L[y_1] + L[y_2]$ and L is not linear //

However

$$M[u_1+u_2] = \frac{d}{dt}(u_1+u_2) + (u_1+u_2) = \frac{du_1}{dt} + u_1 + \frac{du_2}{dt} + u_2 = M[u_1] + M[u_2]$$

$$M[cu] = \frac{d}{dt}(cu) + cu = c\left[\frac{du}{dt} - u\right] = cM[u]$$

hence M is a linear operator //

(c) Solve $\dfrac{du}{dt} + u = e^{t}$ using integrating factor

$$u = \frac{1}{\mu}\int \mu Q\,dt + \frac{c}{\mu}, \quad \mu = e^{\int dt} = e^{t}, \quad Q = e^{t}$$

$$u = e^{t}\int e^{t} e^{t}\,dt + ce^{t} \Rightarrow u = \frac{1}{2}e^{3t} + ce^{t}$$

Hence $\ln y = \frac{1}{2}e^{3t} + ce^{t} \Rightarrow y = ce^{e^{t}} + e^{\frac{1}{2}e^{3t}}$ //

(4) (a) $\dfrac{dv}{dt} = -kv \Rightarrow \dfrac{dv}{v} = -kt \Rightarrow v = ce^{-kt}, \quad v(0) = v_0$

Thus $v(t) = v_0 e^{-kt}$ //

(b) $v = v_0 e^{-kt} \Rightarrow \dfrac{dx}{dt} = v_0 e^{-kt} \Rightarrow x = \int v_0 e^{-kt}\,dt + c \Rightarrow$

$$x = -\frac{v_0}{k}e^{-kt} + c, \quad x(0) = x_0 \Rightarrow c = x_0 + \frac{v_0}{k}$$

Thus $x = \left(x_0 + \dfrac{v_0}{k}\right) - \dfrac{v_0}{k}e^{-kt}$ //

(c) The body stops when $v(t) = 0$, thus when $t \to \infty$

$$\lim_{t \to \infty} x(t) = x_0 + \frac{v_0}{k}$$ //

(5) (a) Direction field (A) $\longleftrightarrow$ (iv)

Direction field (B) $\longleftrightarrow$ (iii)

Direction field (C) $\longleftrightarrow$ (ii)

Direction field (D) $\longleftrightarrow$ (i)

(b) All are nonlinear.

(c) Direction field (A) : No equilibria.

Direction field (B) : No equilibria

Direction field (C) : $y=0$, unstable

$y=\pi/2$, stable

$y=3\pi/2$ $(-\pi/2)$, stable

Direction field (D) : $y=0$, unstable

$y=1$, stable