APPM 2360, Exam #2
Spring 2008

(1) (a) False

(b) True

(c) True

(d) True

(e) True //

(2) (a) $F(x)\cdot F(y) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x+y & 1 \end{bmatrix} = F(x+y)$ //

Notice $F(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ then $F(x)\cdot F(-x) = F(0) \Rightarrow$

$$F^{-1}(x) = F(-x) = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} \quad //$$

Check $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x-x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ //

(b) $\dfrac{dF}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\dfrac{d(F^{-1})}{dx} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

Thus $\dfrac{dF}{dx} + \dfrac{d(F^{-1})}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \mathbb{0}$ //

(c) $B = \dfrac{dF}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, following the hint

$B^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{0}$

Thus $e^B = I + B + \frac{1}{2}B^2 + \dots + \frac{1}{n!}B^n + \dots$

$= I + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ //

(3) (a) The matrix U has three pivot column

Columns 1, 3 & 4 hence $\text{rank}(A) = 3$. //

(b) The system reads

$$2x_1 + x_2 - x_5 = 0$$
$$-x_3 + 3x_5 = 0$$
$$x_4 + x_5 = 0$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

Set $x_2 = s$, $x_5 = r$ then
$$\begin{cases} 2x_1 + s - r = 0 \Rightarrow x_1 = \frac{1}{2}(r-s) \\ x_3 = 3r \\ x_4 = -r \end{cases}$$

Thus
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(r-s) \\ s \\ 3r \\ -r \\ r \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} r + \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s$$

and a basis of the solution space is

$$\left\{ \begin{bmatrix} 1/2 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} //$$

(c) The matrix is rank deficient hence the equation $A\vec{x} = \vec{b}$ doesn't have a solution for every $\vec{b} \in \mathbb{R}^4$. Recall A is 4×5 and $\mathrm{rank}(A) = 3$. //

(4)(a) Define $A = \begin{bmatrix} \lambda & 4 \\ 1 & \lambda \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 4 \\ 0 & \lambda \end{bmatrix}$, $A_2 = \begin{bmatrix} \lambda & 0 \\ 1 & 0 \end{bmatrix}$

clearly $|A| = \lambda^2 - 4$, $|A_1| = |A_2| = 0$.

Unique solution means $|A| \neq 0 \Rightarrow \lambda^2 - 4 \neq 0 \Rightarrow \lambda \neq \pm 2$ //

The unique solution is $x = y = \dfrac{|A_1|}{|A|} = \dfrac{|A_2|}{|A|} = 0$ //

(b) If $z^2 - \lambda z + 1 = 0$ has a double root then

$\Delta = b^2 - 4ac = \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$

Then $|A| = 0$ and the system has infinitely many solutions //

(c) The double root is $z_0 = -\dfrac{\lambda}{2}$ then

$x_0^2 + \left(1 - 5\dfrac{\lambda^2}{4}\right) y_0^2 = 0 \Rightarrow x_0^2 + (1-5)\, y_0^2 = 0 \Rightarrow$

$x_0^2 = 4y_0^2 \Rightarrow \begin{cases} x_0 = -2y_0 \\ x_0 = 2y_0 \end{cases}$

when $\lambda = 2$ the system is reduced to $x_0 = -2y_0$ //

when $\lambda = -2$ the system is reduced to $x_0 = 2y_0$ //

(5) (a) (i) $\begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{bmatrix} =$

$= \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & b_1+b_2 & 0 \\ 0 & 0 & c_1+c_2 \end{bmatrix} \in \mathbb{W}$ hence it is closed under addition

(ii) $k \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & 0 & 0 \\ 0 & kb_1 & 0 \\ 0 & 0 & kc_1 \end{bmatrix} \in \mathbb{W}$ hence it is closed under scalar multiplication

Thus $\mathbb{W}$ is a subspace of $\mathbb{V}$. //

(b) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$

$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= aA_1 + bA_2 + cA_3$

Since A_1, A_2 & A_3 are linearly independent and all A can written as a linear combination of A_1, A_2, A_3 the matrices form a basis for $\mathbb{W}$. //

Clearly $\dim(\mathbb{W}) = 3$ //

(c) $A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= A_1 + 2A_2$

A_1, A_2 and A_4 are not linearly independent and since $\dim(W) = 3$ they do not form a basis //

<u>Extra credit</u>: Since A is not invertible $|A| = 0 \Rightarrow$

$-a^2 - bc = 0 \Rightarrow a^2 + bc = 0$

$A^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix}$

$= (a^2+bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (a^2+bc)\, I$

If $a^2 + bc = 0 \Rightarrow A^2 = 0 \cdot I = \mathbb{O}$ and $A^n \equiv \mathbb{O} \quad n > 1$

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